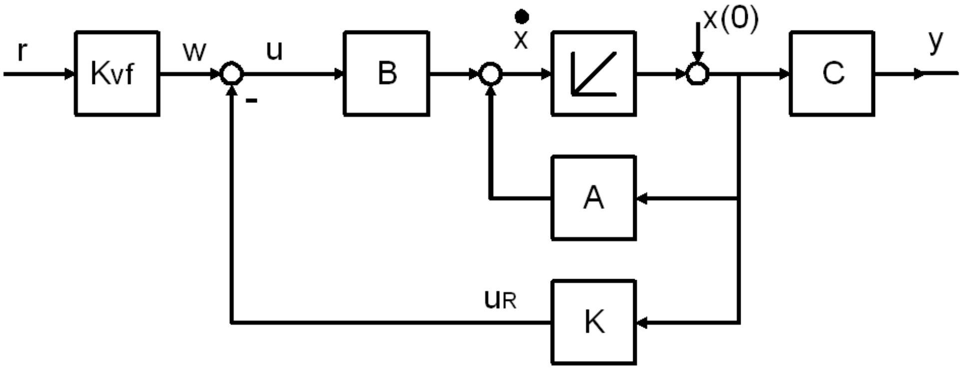
MASTER OF SCIENCE
IN ENGINEERING

Advanced Control

State Regulator



Scope: design of controllers using pole placement and LQ design rules

Keywords: pole placement,
optimal control, LQ regulator, weighting matrices

Prerequisites: state space description

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0 Preliminary Remarks

The path to be followed in this and the next document is the following:

- The first point to be discussed concerns the design of regulators which make use of the state vector x to control a plant. It is assumed that the plant is controllable.

- The next point concerns the estimation of the state vector x in case that it is not completely measured. To design such estimators (observers) it is assumed that the plant is observable.

- The third theme concerns the cooperation between the regulator and the observer. Instead of using the 'true' state-vector, the regulator then utilizes the estimated state-vector. Assumed that the two separate designs of regulator and observer met their specific requirements — are these requirements still fulfilled if the regulator and the observer are used together?

The assumptions about the plant to be controlled are as follows:

- Preferably, the plant is controllable and observable.

- If the plant lacks one or both of the above-mentioned properties (see the examples in Fig. 1), then the non-controllable or non-observable parts can be removed from the model — our methods will work for the remaining model which is controllable and observable.

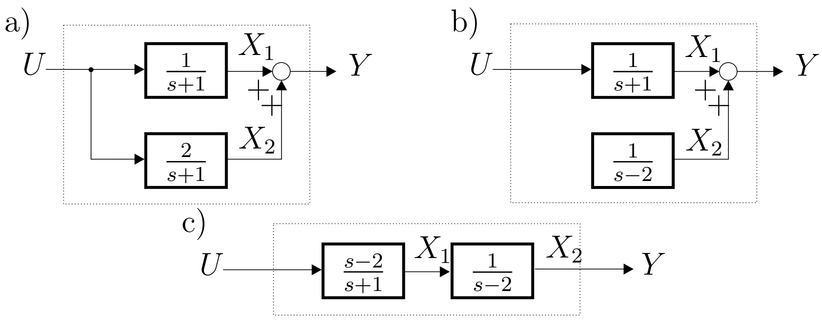

Figure 1: Examples of systems that are not controllable

If the removed parts are e.g. unstable or badly damped then the control system will not be very successful. In that case, additional actuators or sensors are needed.

- The plant is assumed to be a low-pass system with a feed-through matrix $D = 0$. The case $D \neq 0$ is not difficult — it just produces larger equations and diagrams.

In the sequel, the plant often is given as a SISO-system. Of course, that is not a restriction. In principle, it is easier to control a plant with more sensors and more actuators.

1 Feedback of the Full State Vector

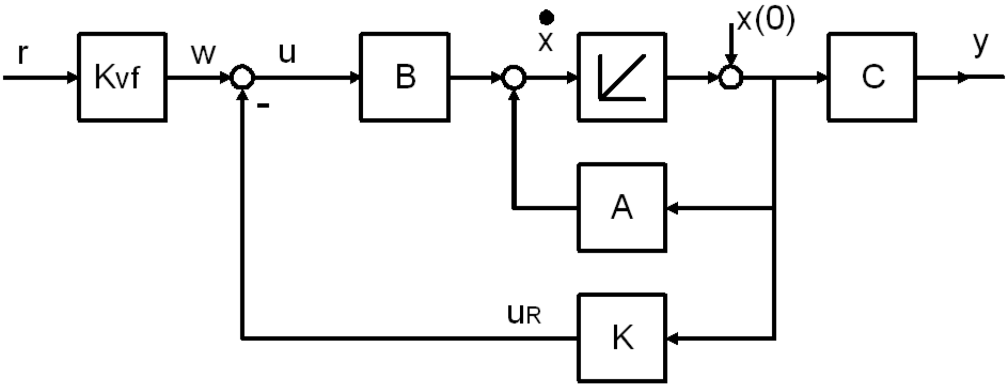

Figure 2: Control structure with feedforward term and full state feedback

In Fig. 2, the control system has two components: the feedback part K, and the feedforward part K_{vf}. The state feedback K is used to determine the dynamics of the closed-loop system, while the prefilter K_{vf} is used to guarantee the static gain $(y_\infty = r_\infty)$[1].
Note that the error signal $e = r - y$ which is used in classical output feedback does not appear in this control structure.

In the feedback signal u_R, the state variables $x_1(t), x_2(t) \ldots x_n(t)$ are weighted by the coefficients $k_1, k_2, \ldots, k_n$.

$$u_R = k_1x_1 + k_2x_2 + ... + k_nx_n = \begin{bmatrix} k_1 & k_2 & ... & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = Kx$$

The control signal u is

$$u = w - u_R = K_{vf} \cdot r - K \cdot x$$

The matrix equations are

$$\dot{x} = A \cdot x + B \cdot (K_{vf} \cdot r - Kx) = A \cdot x - BKx + BK_{vf} \cdot r$$

$$\dot{x} = (A - BK) \cdot x + BK_{vf} \cdot r \quad \text{and} \quad y = C \cdot x$$

The whole system can be described by a new set of matrices A_g, B_g, C_g

$$A_g = A - BK \quad , \quad B_g = BK_{vf} \quad , \quad C_g = C$$

[1] The prefilter, as used here, is just a gain. More sophisticated prefilters might be used to shape the *dynamics* of $Y(s)/R(s)$.

The I/O-description becomes

$$G(s) = \frac{Y(s)}{R(s)} = C_g(sI - A_g)^{-1}B_g = C(sI - A + BK)^{-1}BK_{vf}$$

For the moment we assume that K is known. Then the static gain can be adjusted by setting $G(0) = 1$ or

$$K_{vf} = \left(C(-A + BK)^{-1}B\right)^{-1} \tag{1}$$

This method is sensitive to parameter variations, so in section 4 we will discuss an alternative which includes integral action.

Sections 2 and 3 deal with the question how to calculate K.

2 Pole Placement

According to Definition B of 'controllability', the controllability of (A, B) ensures that the eigenvalues $\lambda_1 \dots \lambda_n$ of $A_g = A - BK$ can be placed arbitrarily by proper choice of K. The eigenvalues of A_g are given by the roots of $\det(\lambda I - A + BK)$. To achieve some desired eigenvalues $\lambda_1 \dots \lambda_n$ the following polynomials must coincide:

$$\det(\lambda I - A + BK) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n) \tag{2}$$

The question which arises is about the choice of 'good' pole locations. Some considerations are:

- Obviously, the poles must be chosen in the left half plane for stability reasons.
- Transients of the signals should decay towards zero faster than some $C \cdot e^{-\sigma_1}$.
- The transients should be well damped, e.g. $\zeta > 1/\sqrt{2}$.
- Forcing transients to decay fast towards zero needs energy. Therefore, an upper bound $\sigma_2 > \sigma$ is meaningful.

While the placement of poles is simple for low order systems, it gets somehow unsatisfying when handling systems of order e.g. 4 or higher.

3 Optimal Control

An alternative way to determine K makes use of the fundamental idea of optimality. Changing the value of K results in a different behavior of the control system. If the behavior of the control system is judged quantitatively, it is convenient to have a scalar performance index (or cost function) $J = f(K)$. Normally, better performance corresponds to smaller values of J. The optimal value of K then can be found by minimizing J, e.g. by varying K systematically. In control theory, systems that

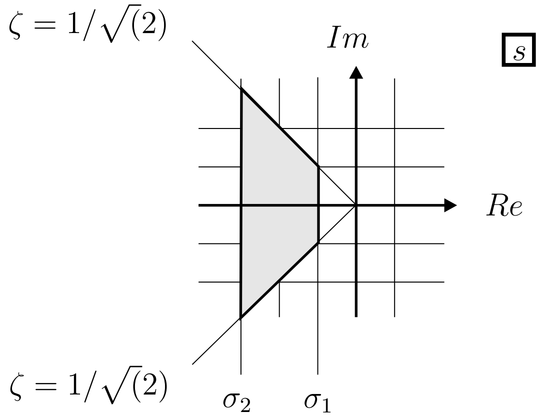

Figure 3: Qualitative sketch of region of 'good' pole locations

are adjusted to provide a minimum performance index are called optimal control systems.

The function f often is defined based on some intermediary variables such as the state vector x or the input u

$$J = f(K) = \int_0^{t_f} g(x,u)\mathrm{d}t$$

where, of course, x and u depend on K. Since it is useless to look at the values of x and u at some specific time only, the index J is formulated as an integral over time. For theoretical considerations, it is convenient to set the final time t_f to infinity. Quadratic terms for the function g are prevalent — on one hand these are meaningful measures, and on the other hand they often yield nice analytical results.

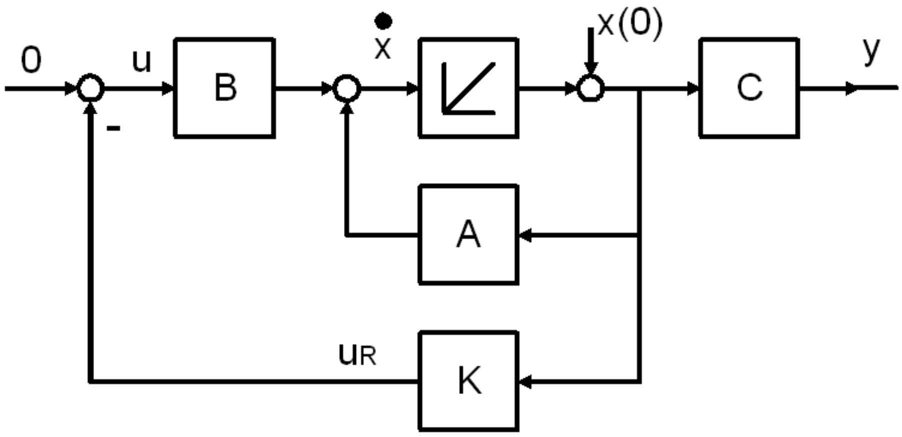

Figure 4: Regulator problem

Consider the following regulator problem, see Fig. 4. The initial state of the system is $x(0) = x_0$. The intention is to take x towards zero 'as fast as possible' by using state feedback. If the norm of x is taken,

$$|x|^2 = {x_1}^2 + {x_2}^2 + ... + {x_n}^2 = \begin{bmatrix} x_1 & x_2 & ... & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x^T x$$

and integrated over time we get the performance index $J = \int\limits_0^{t_f} x^T x \; \mathrm{d}t$. In the following derivation the time horizon is set to infinity, and a more general weighting scheme for the state variables is used with a symmetric, positive semidefinite matrix Q:

$$J = \int\limits_0^\infty x^T Q x \; \mathrm{d}t \tag{3}$$

Note that the general form of the performance index incorporates a term containing u to weigh the control energy. This will be discussed in section 3.1.

To obtain the minimum value of J, we follow [1] and postulate the existence of an exact differential with a constant symmetric matrix P such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^T P x) = -x^T Q x$$

where P is to be determined. Applying the product rule for differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^T P x) = \dot{x}^T P x + x^T \underbrace{\dot{P}}_{0} x + x^T P \dot{x}$$

defining

$$H = A - BK \tag{4}$$

and substituting $\dot{x} = Hx$, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(x^T P x) &= (Hx)^T P x + x^T P (Hx) = x^T H^T P x + x^T P H x \\ &= x^T \underbrace{(H^T P + P H)}_{-Q} x = -x^T Q x \end{aligned} \tag{5}$$

which is the exact differential we are seeking. Substituting that expression in (3) results in the performance index

$$J = \int\limits_0^\infty -\frac{\mathrm{d}}{\mathrm{d}t}(x^T P x)\mathrm{d}t = - \left. x^T P x \right|_0^\infty = -(0 - x^T(0) P x(0)) = x^T(0) P x(0)$$

In the evaluation of the limit at $t = \infty$, we have assumed that the system is stable and hence $x(\infty) = 0$, as desired. Therefore to minimize the performance index J, we consider the two equations:

$$J = \int_0^\infty (x^T Q x)\mathrm{d}t = x^T(0)Px(0) \quad \text{and} \quad (H^T P + PH) = -Q$$

The design steps are then as follows:

1. Determine the matrix P, where H and Q are assumed to be known.
2. Minimize $J = x^T(0)Px(0)$ by adjusting one ore more unspecified system parameters.

3.1 Linear Quadratic Regulator (LQR)

In a more general case, we also consider the control energy of u. A method suitable for computer calculation is stated without proof in the following. Consider the uncompensated MIMO system

$$\dot{x} = Ax + Bu$$

with feedback

$$u = -Kx.$$

The performance index is

$$J = \int_0^\infty (x^T Q x + u^T R u)\mathrm{d}t \tag{6}$$

With Q and R both positive definite, it can be shown that (6) is minimized with

$$K = R^{-1}B^T P \tag{7}$$

To calculate the symmetric $n \times n$ matrix P, an algebraic matrix Riccati (ARE) equation must be solved:

$$A^T P + PA - PBR^{-1}B^T P = -Q.$$

Since the ARE is a quadratic equation, it has multiple solutions. Equation (6) demands for the (only) solution P which is positive definite. The resulting optimal controller is called the linear quadratic regulator (MATLAB command: `lqr`).

The condition about Q can be relaxed: if $Q = \tilde{C}^T\tilde{C} \geq 0$ is positive semidefinite and if $(A, \tilde{C})$ is observable, then P and K can be calculated in the same way. These conditions guarantee that x is observable through the 'virtual' output $\tilde{y} = \tilde{C}x$. In case that $Q \geq 0$ (instead of $Q > 0$) equation (6) delivers at least a solution $P \geq 0$.

3.2 Robustness of the Linear Quadratic Regulator

If the control loop is opened at the input u of the plant (see Fig. 4) then the open-loop transfer function is given by $G_0(s) = K(sI - A)^{-1}B$. In the case of a SI-system the following inequality holds:

$$|1 + G_0(j\omega)| \geq 1 \qquad \text{Kalman inequality}$$

As a consequence, the Nyquist curve of $G_0(j\omega)$ does not enter the unit circle with centre at -1. This results in a phase margin of $\pm 60°$ and a gain margin of $[0.5 \ldots \infty[$. Both margins are minimal guaranteed values. They make the LQ-regulator a considerably robust controller.

In the case of a MI-system it can be shown, that a similar result holds if the Matrix R is chosen as $R = \rho I$ with $\rho > 0$. The above-mentioned margins then are valid for each channel.

4 State Regulator Including Integral Part

In this section we discuss the problem of designing a compensator that provides an asymptotic tracking of a constant reference input $r(t) = r_0$ with zero steady-state error. From classical control it is known that this can be achieved if the open-loop is at least of 'type 1'. Fig. 5 illustrates a possible structure of such a controller including an 'open' integrator.

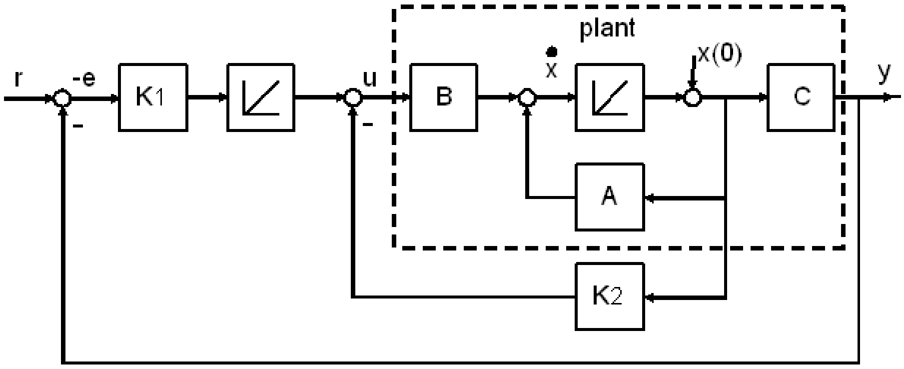

Figure 5: Block diagram of a regulator with K_1 containing an integral part

If K_1 and K_2 are regarded as cascaded controllers then their design can be performed in two steps:

- A 'faster' inner loop is designed by determining K_2, e.g. by means of LQR or pole placement.
- K_1, e.g. a PI-controller, is then tuned using classical methods, resulting in a 'slower' outer loop.

Obviously, in this method the optimality of the inner loop may be destroyed by K_1. To overcome this disadvantage a 'monolithic' controller design is presented, which delivers K_1 and K_2 at once. The idea is formalized here by introducing an internal model of the reference input in the compensator. In Fig. 5 the tracking error e is defined as

$$e = y - r, \qquad \text{(note the sign!)}$$

and its time derivative yields

$$\dot{e} = \dot{y} - \overbrace{\dot{r}}^{0} = \dot{y} = C\dot{x}$$

Defining two intermediate variables, z and w, as

$$z = \dot{x} \ \text{ and } \ w = \dot{u},$$

a new system results:

$$\overbrace{\begin{bmatrix} \dot{e} \\ \dot{z} \end{bmatrix}}^{\dot{\hat{x}}} = \overbrace{\begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}}^{\hat{A}} \overbrace{\begin{bmatrix} e \\ z \end{bmatrix}}^{\hat{x}} + \overbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}^{\hat{B}} \overbrace{w}^{\hat{u}} \tag{8}$$

If the system $(\hat{A}, \hat{B})$ is controllable, then some feedback gain $\hat{K}$ which stabilizes (8) can be designed e.g. by pole placement or using the LQR method

$$\hat{u} = -\hat{K}\hat{x}.$$

Since the error e is a state variable it converges towards zero, i.e. the requirement of an asymptotic tracking with zero steady-state error is fulfilled. Splitting the feedback term yields

$$\hat{u} = w = -\hat{K} \cdot \hat{x} = -[K_1 \quad K_2] \begin{bmatrix} e \\ z \end{bmatrix} = -K_1 e - K_2 z$$

and by integration the final control law

$$u(t) = -K_1 \int_0^t e(\tau) d\tau - K_2 x(t)$$

which is illustrated in Fig. 5.

Bibliography

[1] Richard C. Dorf and Robert H. Bishop. *Modern Control Systems*. Pearson, 2008.

[2] William S. Levine, editor. *The Control Handbook*. The Electrical Engineering Handbook Series. CRC Press/IEEE Press, 1996.

Appendix A

Exercises

Exercise 1. Pole placement I
Given is a system in state space description

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 8 \end{bmatrix}, \; C = \begin{bmatrix} 6.4 & 0 \end{bmatrix}, \; D = 0$$

For each of the following cases, find a controller $u = -k \cdot x + k_{vf} \cdot r$ such that the closed loop system has a unitary static gain and the poles p_1 and p_2.

1. $p_1 = p_2 = -15$
2. $p_{1,2} = -15 \pm 15j$
3. $p_1 = p_2 = -50$

Exercise 2. Pole placement II
Given is a system in state space description

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ b \end{bmatrix}.$$

1. Determine the poles of the system.
2. For which values b is the system not controllable?
3. Determine the feedback gains k_1, k_2 for arbitrary closed loop poles p_1 and p_2.
4. Which restriction does the pole-placement have if b is chosen such that the system is not controllable?

Exercise 3. Optimal control I

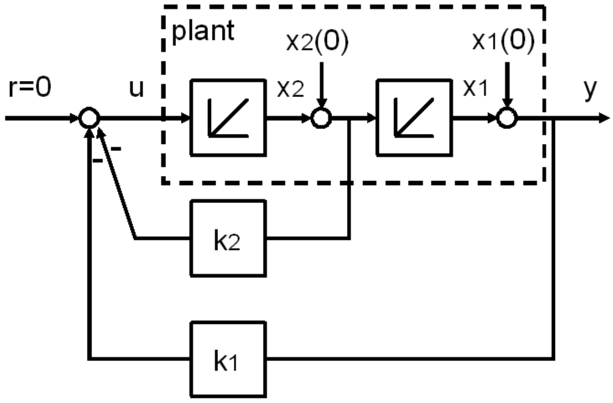

Figure A.1: Block diagram of a LQ regulator

Consider the control system shown in figure A.1. The state vector is $x = [x_1 \quad x_2]^T$. The open-loop system is unstable, therefore state feedback is introduced.

The performance index is

$$J = \int_0^\infty (x^T Q x) \mathrm{d}t.$$

Additional conditions are $Q = I$, $k_1 = k_2 = k$ and $|u| \leq 10$ for the initial conditions $x(0) = [1 \quad 0]^T$.

1. Find the plant's matrices A, B, C and D.
2. Find the matrices H and P.
3. Find k by minimizing the performance index J.

Exercise 4. Optimal control II

Given is a scalar system

$$\dot{x} = ax + bu$$

with a simplified algebraic Riccati equation: $2ap - \frac{1}{r}b^2p^2 + q = 0$.

1. Find a controller $u = -kx$ by minimizing the performance index J

$$J = \int_0^\infty (qx^2 + ru^2) \mathrm{d}t, \quad q > 0, \ r > 0.$$

2. Find a controller $u = -kx$ which stabilizes the system using minimal control energy ($q = 0$).

Exercise 5. LQ-regulator of an inverted pendulum

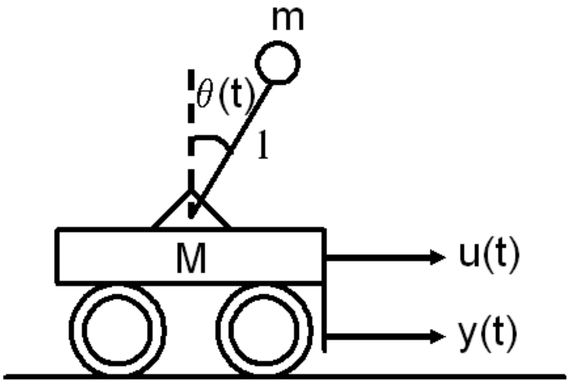

Figure A.2: Inverted pendulum

Consider the inverted pendulum in figure A.2 where the input signal $u(t)$ is a force applied to the cart and the output signal $y(t)$ is the position of the cart. The state vector $[x_1, x_2, x_3, x_4]^T$ is assumed to be $[y, \dot{y}, \theta, \dot{\theta}]^T$. The parameters are given by $M = 1\text{kg}$, $m = 0.1\text{kg}$, $g = 10\text{m/s}^2$ and $l = 1\text{m}$.

The following linearized system description can be found:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \; D = 0.$$

1. Compute some optimal control regulators using the Matlab command `lqr` and plot step responses of the closed-loop systems at different weights of control energy r. Find also suitable prefilters K_{vf}.

2. For the same weights r of the control energy, plot the Bode diagram of the closed-loop systems and discuss the results.

3. Discuss the robustness of the system.

Appendix B

Solutions of Exercises

Solution of Exercise 1. Pole placement I
From equation (2) we get

$$
\begin{aligned}
\det\left(\lambda I - A + BK\right) &= (\lambda - p_1)(\lambda - p_2) \\
\det\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}0 & 1\\ 0 & -4\end{bmatrix} + \begin{bmatrix}0 & 0\\ 8k_1 & 8k_2\end{bmatrix}\right) &= \\
\det\left(\begin{bmatrix}\lambda & -1\\ 8k_1 & \lambda + 4 + 8k_2\end{bmatrix}\right) &= \\
\lambda\left(\lambda + 4 + 8k_2\right) + 8k_1 &= \\
\lambda^2 + \left(4 + 8k_2\right)\lambda + 8k_1 &= \lambda^2 - (p_1 + p_2)\lambda + p_1 p_2 \\
&\Updownarrow \\
p_1 p_2 = 8k_1 \quad \Leftrightarrow \quad k_1 &= \frac{p_1 p_2}{8} \\
-(p_1 + p_2) = 4 + 8k_2 \quad \Leftrightarrow \quad k_2 &= -\frac{p_1 + p_2 + 4}{8}.
\end{aligned}
$$

Moving the closed-loop poles p_1 and p_2 to the left results in faster system dynamics as well as in higher gains k_1 and k_2. Possible pole locations are limited due to physical constraints on the magnitude of the input $u = -kx$.

The prefilter k_{vf} is calculated with equation (1).

$$
\begin{aligned}
k_{vf} &= \left(C\left(-A + BK\right)\right)^{-1} B + D\Big)^{-1} \\
&= \left(\begin{bmatrix}6.4 & 0\end{bmatrix} \frac{\begin{bmatrix}4 + 8k_2 & 1\\ -8k_1 & 0\end{bmatrix}}{8k_1} \begin{bmatrix}0\\ 8\end{bmatrix}\right)^{-1} \\
&= \left(\frac{6.4 \cdot 8}{8k_1}\right)^{-1} \\
&= \frac{k_1}{6.4}
\end{aligned}
$$

1.

$$
\begin{aligned}
k_1 &= \frac{(-15)(-15)}{8} = 28.125 \\
k_2 &= -\frac{-15-15+4}{8} = 3.25 \\
k_{vf} &= \frac{28.125}{6.4} = 4.39453125
\end{aligned}
$$

2.

$$
\begin{aligned}
k_1 &= \frac{(-15+15j)(-15-15j)}{8} = 56.25 \\
k_2 &= -\frac{-15+15j-15-15j+4}{8} = 3.25 \\
k_{vf} &= \frac{56.25}{6.4} = 8.7890625
\end{aligned}
$$

3.

$$
\begin{aligned}
k_1 &= \frac{(-50)(-50)}{8} = 312.5 \\
k_2 &= -\frac{-50-50+4}{8} = 12 \\
k_{vf} &= \frac{312.5}{6.4} = 48.828125
\end{aligned}
$$

Solution of Exercise 2. Pole placement II

1.

$$
\begin{aligned}
\det(\lambda I - A) &= \det\left(\begin{bmatrix} \lambda+3 & -1 \\ 0 & \lambda+2 \end{bmatrix}\right) = (\lambda+3)(\lambda+2) \\
p_{1/2} &= \{-3, -2\}
\end{aligned}
$$

2.

$$
\begin{aligned}
\det(P_c) &= \det\left(\begin{bmatrix} 1 & -3+b \\ b & -2b \end{bmatrix}\right) \\
&= b - b^2 \\
&= b(1-b)
\end{aligned}
$$

The system is not controllable for $b = 0$ and $b = 1$.

3.

$$
\begin{aligned}
\det(\lambda I - A + BK) &= (\lambda - p_1)(\lambda - p_2) \\
\det\left(\begin{bmatrix} \lambda + 3 + k_1 & -1 + k_2 \\ bk_1 & \lambda + 2 + bk_2 \end{bmatrix}\right) &= \\
\lambda^2 + (k_1 + bk_2 + 5)\lambda + (2+b)k_1 + 3bk_2 + 6 &= \lambda^2 - (p_1 + p_2)\lambda + p_1 p_2 \\
&\Updownarrow \\
-p_1 - p_2 &= k_1 + bk_2 + 5 \\
p_1 p_2 &= (2+b)k_1 + 3bk_2 + 6 \\
&\Updownarrow \\
\begin{bmatrix} -p_1 - p_2 - 5 \\ p_1 p_2 - 6 \end{bmatrix} &= \begin{bmatrix} 1 & b \\ 2+b & 3b \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \qquad \text{(B.1)} \\
\frac{1}{b(1-b)} \begin{bmatrix} 3b & -b \\ -2-b & 1 \end{bmatrix} \begin{bmatrix} -p_1 - p_2 - 5 \\ p_1 p_2 - 6 \end{bmatrix} &= \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \\
&\Updownarrow \\
k_1 &= \frac{(p_1+3)(p_2+3)}{b-1} \\
k_2 &= \frac{(p_1+2)(p_2+2)}{b} - \frac{(p_1+3)(p_2+3)}{b-1}
\end{aligned}
$$

4. k_1 and k_2 can not be evaluated for the uncontrollable cases (divisions by zero). There are two exceptions when we go back to equation (B.1).

   Case $b = 0$:

$$
\begin{bmatrix} -p_1 - p_2 - 5 \\ p_1 p_2 - 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} k_1
$$

   k_2 does not influence the poles and can be selected arbitrarily. To fulfill the matrix equation, either $p_1 = -2$ or $p_2 = -2$ has to be selected. The equation reduces then to

$$
k_1 = -p - 3.
$$

   Case $b = 1$:

$$
\begin{bmatrix} -p_1 - p_2 - 5 \\ p_1 p_2 - 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}
$$

   The matrix equation is only fulfilled if either $p_1 = -3$ or $p_2 = -3$. The equation reduces then to

$$
-p - 2 = k_1 + k_2.
$$

In a non-controllable system, at least one pole(s) cannot be moved by state-feedback. Pole placement then can be performed only if this pole / these poles remain in the list of the desired closed-loop poles. The feedback gain k is then not unique.

As an alternative, a minimal system representation can be used to place the poles. (That involves reduction of the system order.)

Solution of Exercise 3. Optimal control I

1. Read out the equations directly from the block diagram or write down the transfer function $G(s) = \frac{1}{s^2}$ and use the controllable canonical form.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} u$$
$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\end{bmatrix} u$$

2. The solution can be found using equations (4) and (5):

$$\begin{aligned} H &= A - BK \\ &= \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}k & k\end{bmatrix} \\ &= \begin{bmatrix}0 & 1\\ -k & -k\end{bmatrix}\end{aligned}$$

$$H^T P + PH = -Q$$

It follows from $P = P^T$ that $p_{12} = p_{21}$:

$$\begin{bmatrix}0 & -k\\ 1 & -k\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix} + \begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}0 & 1\\ -k & -k\end{bmatrix} = -\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}-kp_{12} & -kp_{22}\\ p_{11} - kp_{12} & p_{12} - kp_{22}\end{bmatrix} + \begin{bmatrix}-kp_{12} & p_{11} - kp_{12}\\ -kp_{22} & p_{12} - kp_{22}\end{bmatrix} = -\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}2kp_{12} & -p_{11} + kp_{12} + kp_{22}\\ -p_{11} + kp_{12} + kp_{22} & -2p_{12} + 2kp_{22}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\Updownarrow$$
$$\begin{aligned} 2kp_{12} = 1 \quad &\Leftrightarrow \quad p_{12} = \frac{1}{2k} \\ -2p_{12} + 2kp_{22} = 1 \quad &\Leftrightarrow \quad p_{22} = \frac{k+1}{2k^2} \\ -p_{11} + kp_{12} + kp_{22} = 0 \quad &\Leftrightarrow \quad p_{11} = \frac{2k+1}{2k}\end{aligned}$$

3.

$$\begin{aligned} J(k) &= x^T(0)Px(0) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= p_{11} = \frac{2k+1}{2k} = 1 + \frac{1}{2k} \end{aligned}$$

Because P has to be positive definite ($P > 0$), k has to be positive ($k > 0$). You can verify this condition by checking the eigenvalues of P which have to be positive.

$J(k)$ is monotonically decreasing for $k > 0$. The optimum is achieved for the highest possible k. Because we have the additional condition $|u| \leq 10$, k is bounded by

$$\begin{aligned} u(0) = Kx(0) = \begin{bmatrix} k & k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = k \leq 10 \\ \Rightarrow k_{opt} = 10. \end{aligned}$$

Solution of Exercise 4. Optimal control II

1. Solving the Riccati equation

$$\left(-\frac{1}{r}b^2\right)p^2 + 2ap + q = 0$$

results in the two solutions

$$p = \frac{-2a \pm \sqrt{4a^2 + 4\frac{1}{r}b^2 q}}{-2\frac{1}{r}b^2} = \frac{r}{b^2}\left(a \mp \sqrt{a^2 + b^2\frac{q}{r}}\right).$$

We are only interested in the positive solution. Because $q \geq 0$ and $r > 0$

$$b^2\frac{q}{r} \geq 0 \qquad \Rightarrow \qquad a \leq \sqrt{a^2 + b^2\frac{q}{r}}.$$

The positive solution is

$$p = \frac{r}{b^2}\left(a + \sqrt{a^2 + b^2\frac{q}{r}}\right).$$

The optimal feedback gain k according to equation (7) is

$$k = r^{-1}bp = \frac{1}{b}\left(a + \sqrt{a^2 + b^2\frac{q}{r}}\right).$$

2. If $q \to 0$, the Riccati equation becomes

$$-\frac{1}{r}b^2p^2 + 2ap = \left(-\frac{1}{r}b^2p + 2a\right)p = 0$$

with the two solutions

$$p_1 = \frac{2ar}{b^2}, \quad p_2 = 0.$$

It follows that the solution p of the Riccati equation is

$$p = \begin{cases} 0 & \text{if } a \le 0 \\ \frac{2ar}{b^2} & \text{if } a > 0 \end{cases}.$$

The feedback gain $k = r^{-1}bp$ is then

$$k = \begin{cases} 0 & \text{if } a \le 0 \\ \frac{2a}{b} & \text{if } a > 0 \end{cases}.$$

The interpretation of this result is: if the system is stable ($a \le 0$), then we need no control input to stabilize the system (the system operates in open loop). For an unstable system, the feedback k moves the unstable pole $a > 0$ to

$$a - bk = a - b\frac{2a}{b} = -a.$$

Solution of Exercise 5. LQ-regulator of an inverted pendulum

1.

Listing B.1: Calculate LQ-regulators and plot step responses

```
g = 10;
l = 1;
M = 1;
m = 0.1;

A = [0 1 0 0;
     0 0 -m*g/M 0;
     0 0 0 1;
     0 0 (M+m)*g/(M*l) 0];
B = [0; 1/M; 0; -1/(M*l)];
C = [1 0 0 0];

Q = eye(size(A));
R = logspace(-2,2,5);

L = cell(1,length(R));
G = cell(1,length(R));
```

```
for i = 1:length(R)
    K = lqr(A,B,Q,R(i));
    kvf = 1/(C/(-A+B*K)*B);
    G{i} = ss(A-B*K,B,C,[])*kvf;
    L{i} = ss(A,B,K,[]);
end

figure(1);
step(G{:});
legends = cell(1,length(R));
for i = 1:length(R)
    legends{i} = ['R = ' num2str(R(i),'%g')];
end
legend(legends,'Location','East');
```

Refer to figure B.1 for the step responses.

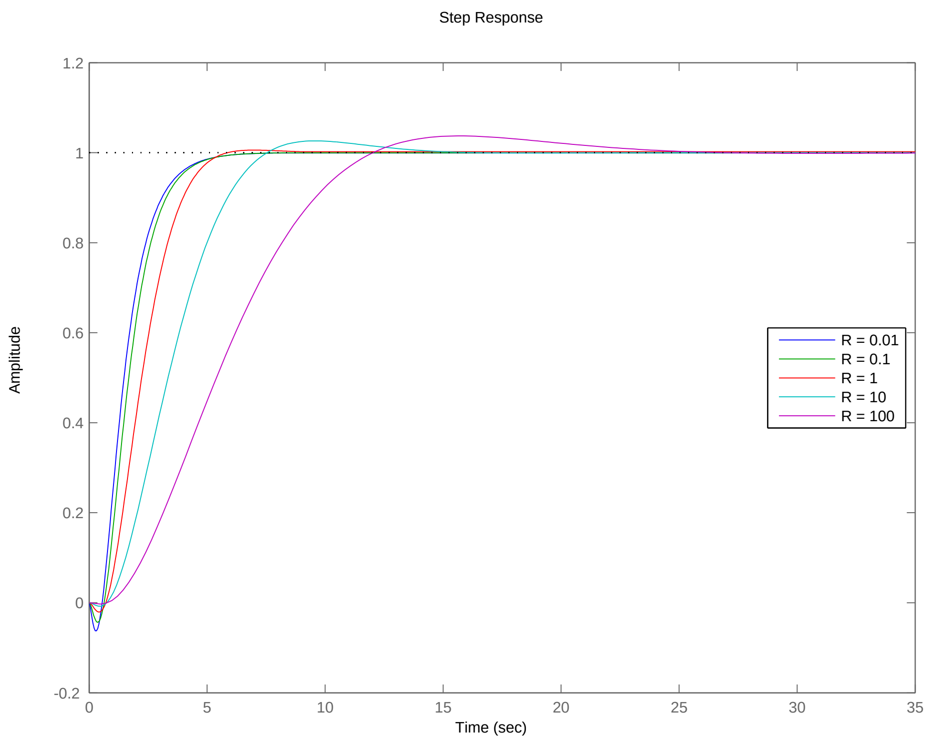

Figure B.1: Step responses

2.

Listing B.2: Create Bode diagrams

```
figure(2);
bode(G{:});
grid on;
legend(legends,'Location','East');
```

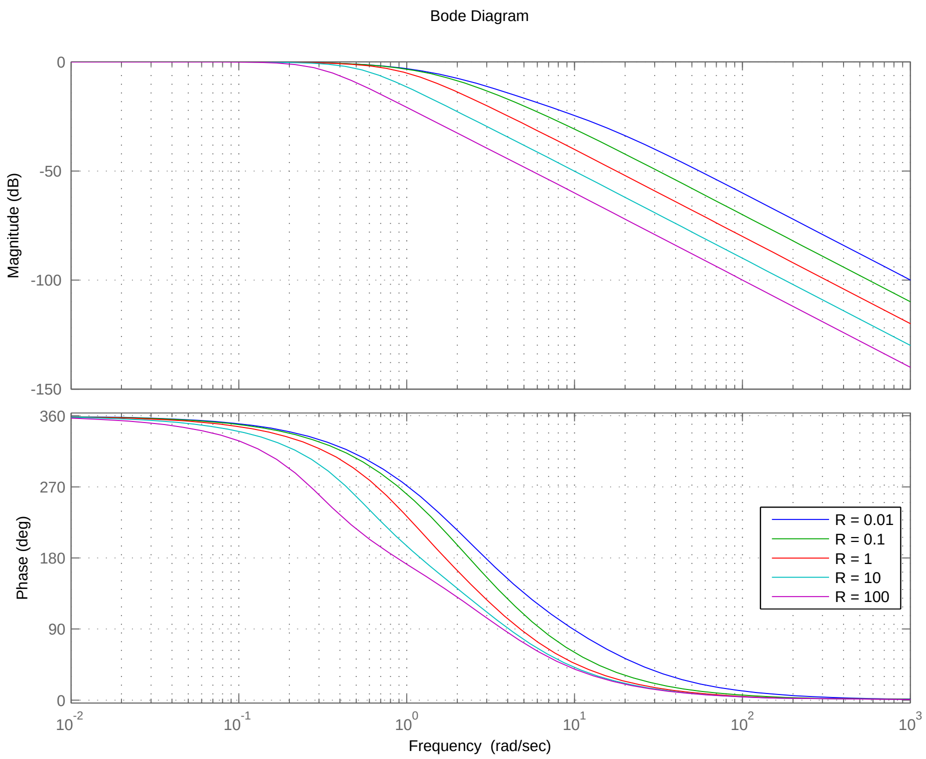

Figure B.2: Bode diagrams

Decreasing the weight r for the input u leads to a closed-loop system with a faster response. This can be seen in the step responses in figure B.1 and in the increased bandwidth in the Bode diagrams in figure B.2.

3. The robustness of a system can be determined in the nyquist plot of the open loop system. One can verify that none of the plots in figure B.3 enter the unit circle around -1. This means that all loops have a phase margin $\phi_m > 60°$ and a gain margin $\frac{1}{2} < G_m < \infty$.

Listing B.3: Create Nyquist plots

```
figure(3);
phi = (0:100)*2*pi/100;
nyquist(L{:});
hold on;
plot(cos(phi)-1,sin(phi),'k');
hold off;
axis([-6 10 -6 6]);
axis equal;
legend(legends,'Location','East');
```

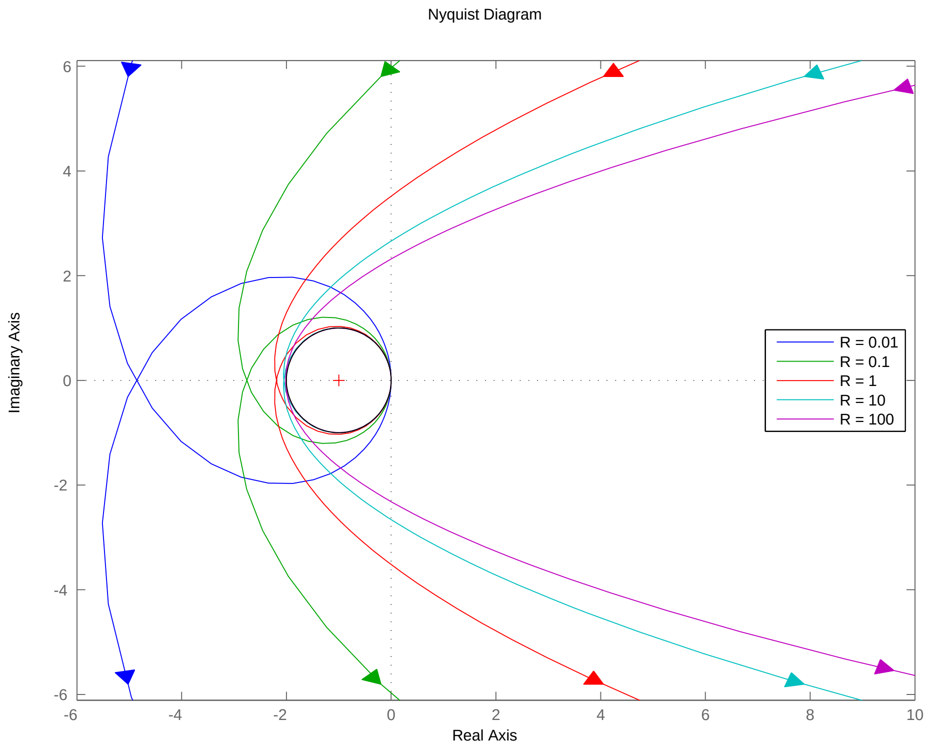

Figure B.3: Nyquist plots