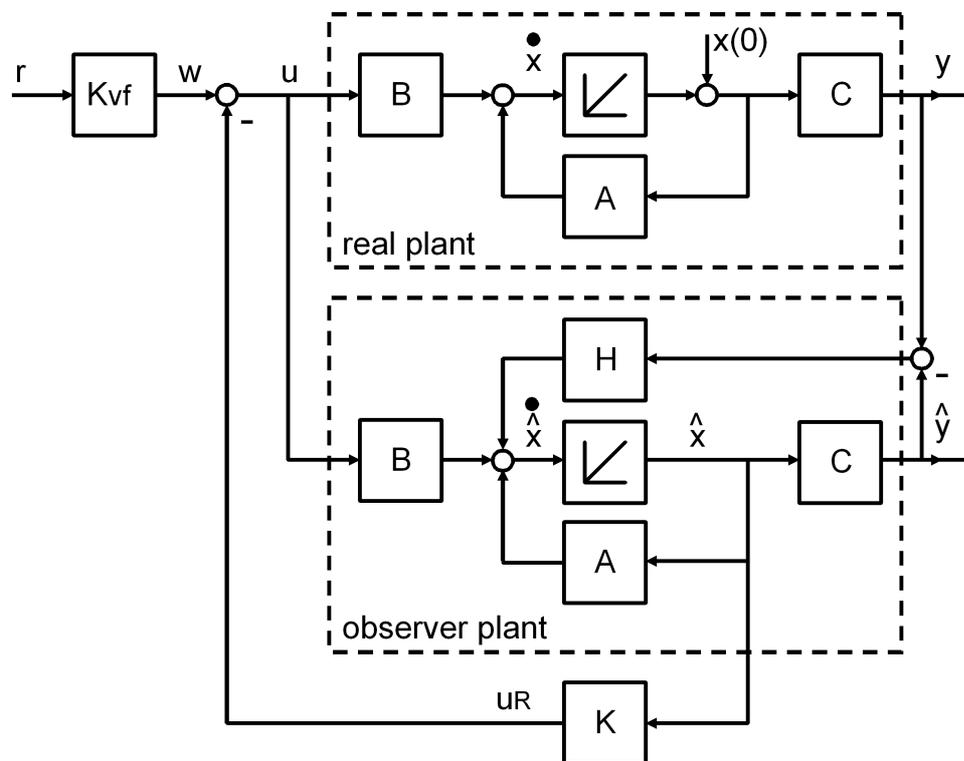


## Advanced Control State Regulator with Observer



Scope	design of observers and increasing the robustness of state regulators with observers (loop transfer recovery)
Keywords	observer, separation principle, linear quadratic gaussian (LQG), loop transfer recovery (LTR), robustness
Prerequisites	state space description, state regulator
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Date	September 16, 2011

## 1 Observer

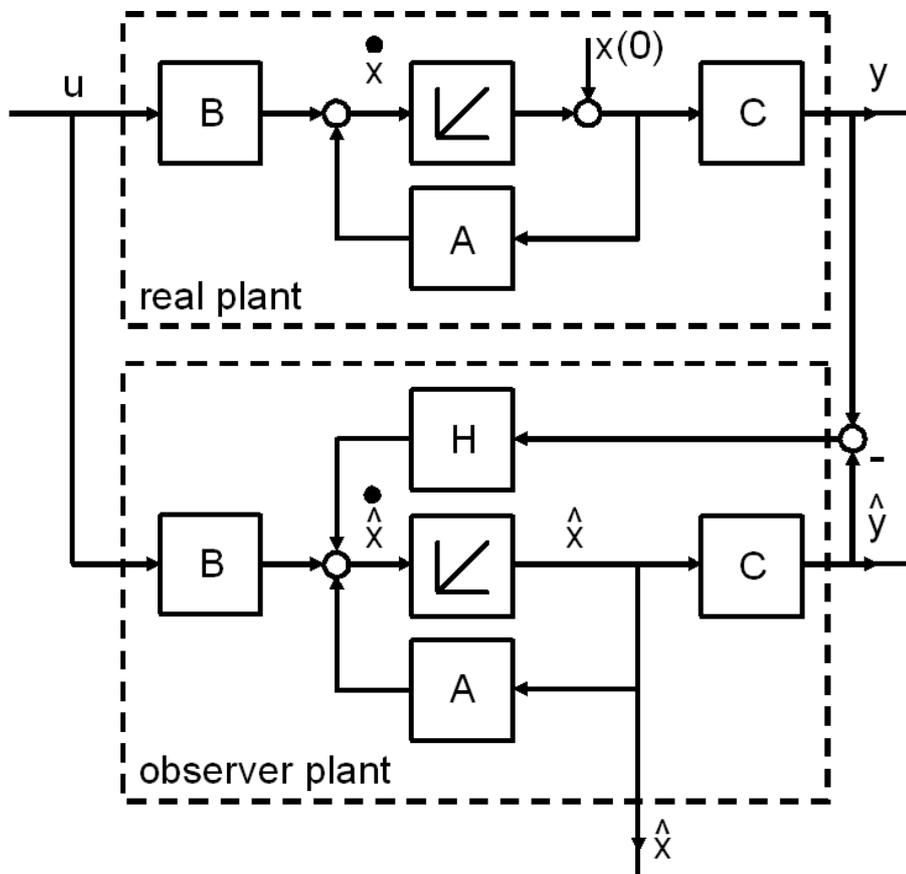


Figure 1: Observer design

The LQ-regulator design assumed that all the state variables are available for feedback. In practice, not all state variables are measured. The reasons are that this is not physically feasible or that the sensors required are probably too expensive. In this section we demonstrate how to reconstruct the complete state information based on the measured output  $y$ . The assumption is that we know the system description  $(A, B, C)$  and that  $(A, C)$  is observable.

One method of estimating the state  $x$  in an observer is to construct a full order model of the plant dynamics

$$\dot{\hat{x}} = A\hat{x} + Bu$$

where  $\hat{x}$  is the estimate of the actual state  $x$ , and  $A$ ,  $B$  and  $u(t)$  are known. If this observer can be started with the correct initial condition  $\hat{x}(0) = x(0)$  it will always deliver  $\hat{x}(t) = x(t)$ . However, it is precisely the lack of information about  $x(0)$  that requires the construction of an observer. The effect of an error in the initial condition  $\hat{x}$  can be studied by defining the estimation error,  $\tilde{x} = x - \hat{x}$ .

$$\begin{aligned}
 \dot{x} &= Ax + Bu & (1) \\
 \dot{\hat{x}} &= A\hat{x} + Bu \\
 \dot{x} - \dot{\hat{x}} &= A(x - \hat{x}) \\
 \dot{\tilde{x}} &= A\tilde{x}
 \end{aligned}$$

The dynamics of  $\tilde{x}$  are described by the system matrix  $A$ . If it is unstable, then the estimation error diverges. If  $A$  is stable then  $\tilde{x}$  converges towards zero — however, probably very slowly. Furthermore, effects like noise or errors in the system description ( $A, B$ ) might cause the estimate to diverge from the true state.

We now introduce feedback in the observer, to enforce stability of the error dynamics and/or for faster convergence. The difference between the measured and the estimated outputs are used to correct the estimated state, see Fig. 1.

The equation for this scheme is

$$\begin{aligned}
 \dot{\hat{x}} &= A\hat{x} + Bu + H(y - C\hat{x}) \\
 &= (A - HC)\hat{x} + Bu + HCx.
 \end{aligned}$$

resulting in the error dynamics

$$\dot{\tilde{x}} = (A - HC)\tilde{x}. \quad (2)$$

If  $H$  is chosen such that the eigenvalues of  $(A - HC)$  have negative real parts then  $\tilde{x}$  will converge towards zero.

Note that in all considerations we have assumed that the plant description is known and that no noise is present. It is important to emphasize that the nature of the plant and of the observer are quite different. The plant is a physical system such as a chemical process or servomechanism, whereas the observer is usually an electronic unit computing the estimated state. In practice, there are differences between the real plant and the plant modelled within the observer, e.g. disturbing inputs, process noise, different  $A, B$  and  $C$  matrices or nonlinear effects. All these effects prevent the estimation error from reaching zero. Normally, if  $H$  is chosen properly, the error can be kept acceptably small.

Before discussing the design of the observer feedback gain  $H$  we have a look at the concept of *duality*. The structures of the controllability matrix and the observability matrix have some obvious similarities. Assume that some MATLAB routine `build_Pc` exists which calculates  $P_c$ , the controllability matrix: `Pc = build_Pc(A,B)`. Instead of implementing the corresponding routine (`build_Po`) the correspondences

expression relevant for regulator  $\iff$  expression relevant for observer

$$P_c \iff P_o^T$$

$$B \iff C^T$$

$$A \iff A^T$$

can be used to ‘recycle’ the existing code: `Po = build_Pc(A',C')`. Or implemented as a function `build_Po` which wraps the existing function `build_Pc`:

```
function Po = build_Po(A,C)
Po = build_Pc(A',C)'
```

## 2 Observer Design using Dual Methods

In the case of the regulator design we discussed the method of pole placement: Given are  $A$  and  $B$  — determine  $K$  such that the eigenvalues of  $A - BK$  coincide with some predefined poles. For the observer design, the problem occurs in its dual form. Given are  $A$  and  $C$  — determine  $H$  such that the eigenvalues of  $A - HC$  coincide with some predefined poles. Note that  $A - HC$  and its transpose,  $A^T - C^T H^T$ , have the same eigenvalues. So, for the pole placement, we add a new entry to the list of correspondences:

$$K \iff H^T$$

The MATLAB functions `acker` and `place` which are implemented to solve the pole placement for the regulator (`K = place(A,B,p)`) can be used for the observer as well: `H = place(A',C',p)`. Note that `p` is a vector containing the desired eigenvalues; it does not matter whether `p` is arranged as column vector or row vector.

What about the LQR-design? In MATLAB some optimal regulator gain  $K$  can be calculated using `K = lqr(A,B,Q,R)`. Obviously, the command `lqr` can be adapted to calculate an observer gain in a dual way: `H = lqr(A',C',Q,R)`.

Due to the duality, the resulting observer is stable and considerably robust. If the observer loop is opened at the output  $\hat{y}$  (see Fig. 1) then the open-loop transfer function, given by  $G_0(s) = C(sI - A)^{-1}H$ , has at least a phase margin of  $\pm 60^\circ$  and a gain margin of  $[0.5 \dots \infty[$  on each channel.

But is the observer also optimal? And if yes, in what sense? — The answer to the first question is positive, and the answer to the second question will be given only in abbreviated form<sup>1</sup>.

Reconsidering the LQR problem, we see that its formulation is entirely deterministic. The calculation of the optimal  $K$  comprises two steps:

- I.  $A^T P + PA - PBR^{-1}B^T P = -Q$  delivers a positive definite solution  $P$
- II.  $K = R^{-1}B^T P$

Surprisingly, the formulation of the corresponding (‘dual’) optimal observer problem has a stochastic background.  $Q$  and  $R$  are used to characterize the process noise

<sup>1</sup>We refer e.g. to [1] or [2], see keywords ‘Kalman filter’ or ‘Kalman-Bucy filter’.

and measurement noise, resp., and the optimal observer gain  $H$  minimizes the expectation value of the covariance matrix of the estimation error:  $E\{\tilde{x} \cdot \tilde{x}^T\}$ . The calculation of the optimal  $H$  for this so-called LQG (Linear Quadratic Gaussian) problem comprises two steps

- I.  $AP + PA^T - PC^T R^{-1} CP = -Q$  delivers a positive definite solution  $P$
- II.  $H = (R^{-1} CP)^T = PC^T R^{-1}$

Clearly, this procedure is dual to the above-mentioned LQR design.

### 3 State Space Regulator in Combination with Observer

So far, we have discussed two independent questions:

1. Regulator design:

- It is possible to stabilize a controllable system by state feedback with a (static) gain  $K$ .
- $K$  can be designed e.g. by pole placement. This results in a stable closed-loop system if the poles are chosen in the left half plane.
- $K$  can be designed more sophisticatedly, e.g. using the LQR method. Again, a stable closed-loop system results. Moreover, there are some robustness guarantees in form of phase and gain margin.

2. Observer design:

- For an observable system, it is possible to build an observer with an estimated state vector  $\hat{x}$  converging towards the true state vector  $x$ .
- The gain  $H$  used in the observer can be designed e.g. by pole placement. The observer is stable if the poles are chosen in the left half plane.
- $H$  can be designed more sophisticatedly, e.g. using the LQG method. Again, a stable closed-loop system results. Moreover, there are some robustness guarantees in form of phase and gain margin.

The motivation behind the development of the observer was to provide the regulator with an estimation  $\hat{x}$  of the true state vector  $x$ . If the observer is combined with then the state feedback the diagram shown in Fig. 2 results.

The separate designs of state feedback and observer guarantee some features such as stability or gain/phase margins. The rest of this section is devoted to the question whether these guarantees still hold if the overall system is considered.

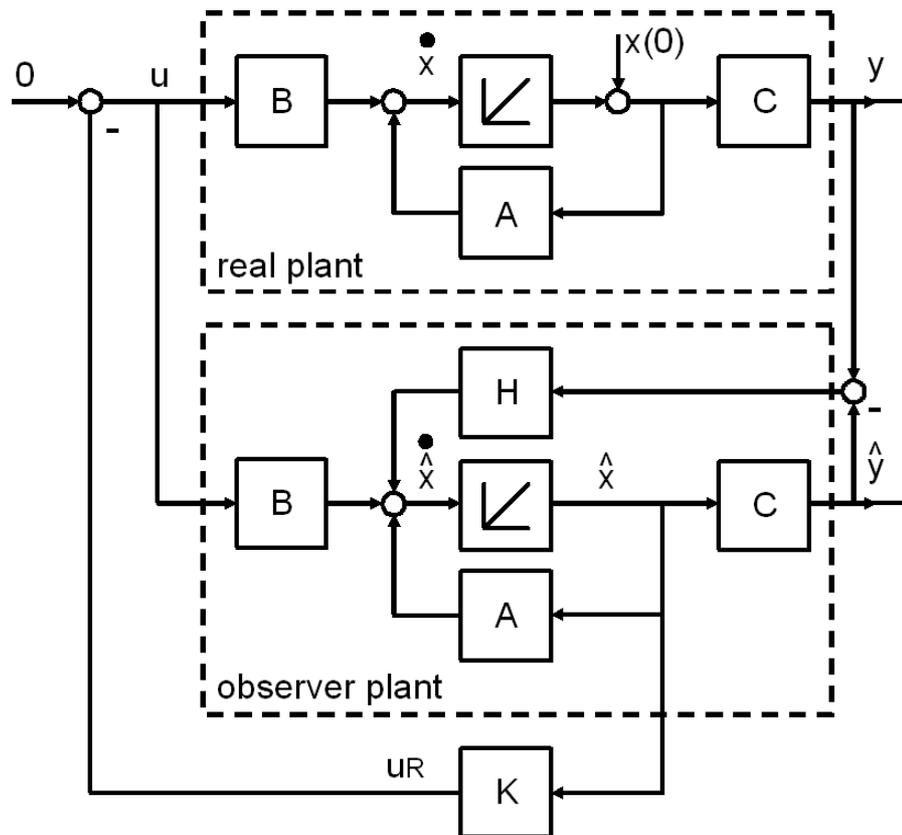


Figure 2: State regulator with observer

### 3.1 Pole Locations, Separation Principle

The plant equation with feedback is now given by

$$\begin{aligned}\dot{x} &= Ax - BK\hat{x} \\ &= Ax - BK(x - \tilde{x}) \\ &= (A - BK)x + BK\tilde{x}\end{aligned}$$

which can be rewritten in terms of the state error  $\tilde{x}$ . The error dynamics (2) are still valid:

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

Combined, this yields the overall system dynamics in state-space form:

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - HC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

The characteristic equation of this closed-loop system is

$$\det \begin{bmatrix} sI - A + BK & -BK \\ 0 & sI - A + HC \end{bmatrix} = 0$$

Because the matrix is block triangular, we can separate two minors

$$\det(sI - A + BK) \cdot \det(sI - A + HC) = 0$$

In other words, the set of poles of the combined system consists of the union of the control poles and the observer poles. This means that the design of the controller and the observer can be carried out independently, yet when they are used together in this way, the poles remain unchanged. This fact is called ‘separation principle’.

### 3.2 Phase and Gain Margins

The block diagram of Fig. 2, slightly rearranged, results in Fig. 3. The control system (consisting of regulator and observer but omitting the feedforward term  $K_{vf}$ ) is a compensator with the description

$$\begin{aligned}\dot{\hat{x}} &= (A - BK - HC)\hat{x} - He \\ u &= -K\hat{x}\end{aligned}$$

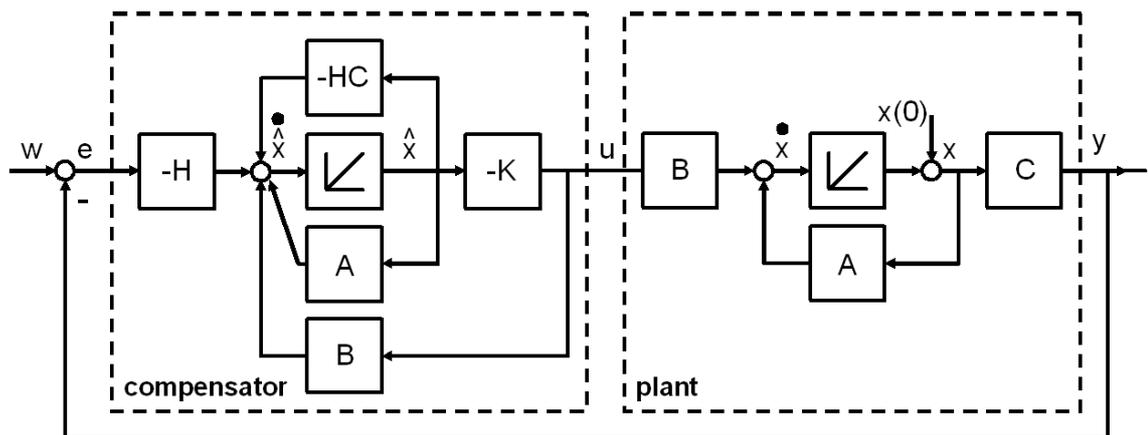


Figure 3: State regulator and observer, drawn as a compensator

The LQR and the LQG designs gave nice results by means of phase and gain margins. As mentioned earlier, the open loops of the regulator ( $G_{0,reg}$ ) and observer ( $G_{0,obs}$ )

$$\begin{aligned}G_{0,reg}(s) &= K(sI - A)^{-1}B \\ G_{0,obs}(s) &= C(sI - A)^{-1}H\end{aligned}$$

have at least a phase margin of  $\pm 60^\circ$  and a gain margin of  $[0.5 \dots \infty[$  on each channel.

According to Fig. 3 the transfer function of the compensator is given by

$$G_c(s) = \frac{U(s)}{E(s)} = K(sI - A + BK + HC)^{-1}H$$

and, with the loop opened at the plant input  $u$ , the open-loop transfer function is

$$G_0(s) = K(sI - A + BK + HC)^{-1}H \cdot C(sI - A)^{-1}B.$$

The obvious question is whether  $G_0(s)$  has any guaranteed margins. Unfortunately, this is not the case. In [3], Doyle gives an example in which the margins can be made arbitrarily small.

## 4 Loop Transfer Recovery (LQG/LTR and LQR/LTR)

If some robustness of the closed loop is desired, the LQR and LQG designs cannot be carried out independently. The following result from matrix theory helps to recover the robustness of the closed loop. Assume that

- the LQR design is performed with arbitrary weighting matrices, resulting in a regulator gain  $K$ ,
- the LQG design is carried out with weighting matrices  $Q = \rho BB^T$  and  $R = I$ , resulting in an observer gain  $H$ .

Then, if the plant  $G(s)$  is minimum-phase, it can be shown that

$$\lim_{\rho \rightarrow \infty} G_0(s) = \lim_{\rho \rightarrow \infty} \underbrace{K(sI - A + BK + HC)^{-1}H}_{G_c(s)} \cdot \underbrace{C(sI - A)^{-1}B}_{G(s)} = \underbrace{K(sI - A)^{-1}B}_{G_{0, reg}(s)}. \quad (3)$$

Remarks:

- Since  $G_0(s)$  can come as close to  $G_{0, reg}(s)$  as desired, this means that the phase and gain margins can come as close to (or beyond) the minimal margins guaranteed by LQR design ( $\pm 60^\circ / [0.5 \dots \infty]$ ). The recovery of the open-loop transfer can be arbitrarily good.
- The steps to calculate the LQG solution are

- I.  $AP + PA^T - PC^T C P = -\rho BB^T \quad \Rightarrow$  choose solution  $P > 0$
- II.  $H = PC^T$

- Using large values of  $\rho$  in the LQG design pretends that the process noise is large, compared to the measurement noise. In tendency, increasing  $\rho$  leads to higher gain in the observer feedback matrix  $H$ , thus resulting in a higher sensitivity for noisy signals.

Clearly, this procedure does not deliver an optimal matrix  $H$  in the original sense of the LQG design, where the ‘real’ values of  $Q$  and  $R$  are to be used.

- If equation (3) is rearranged as

$$\lim_{\rho \rightarrow \infty} G_c(s) = G_{0, reg}(s) \cdot G^{-1}(s)$$

it becomes obvious that in the limit, the compensator design inverts the plant. By increasing  $\rho$ , some of the observer poles move towards the zeros of the plant. Therefore, full loop transfer recovery is not possible for nonminimum-phase plants.

- For reasons of duality, the previously mentioned also holds the other way round. If the loop is opened at the plant output  $y$ , then the open-loop transfer function is

$$\hat{G}_0(s) = C(sI - A)^{-1}B \cdot K(sI - A + BK + HC)^{-1}H.$$

By choosing  $Q = \rho C^T C$  and  $R = I$ , the LQR design can be used to recover the LQG loop transfer:

$$\lim_{\rho \rightarrow \infty} \hat{G}_0(s) = \lim_{\rho \rightarrow \infty} \underbrace{C(sI - A)^{-1}B}_{G(s)} \cdot \underbrace{K(sI - A + BK + HC)^{-1}H}_{G_c(s)} = \underbrace{C(sI - A)^{-1}H}_{G_{0, obs}(s)} \quad (4)$$

- Often, both recovery schemes are termed LQG/LTR. However, strictly speaking, the LQG/LTR method corresponds to (4), where the loop transfer of the LQG design is recovered. The dual design in (3) is the *dual* LQG/LTR method — here, the loop transfer of the LQR design is recovered.

#### 4.1 LTR design using the command `lqr` to determine $K$ and $H$

In sections 4.1 and 4.2 the MATLAB commands `lqr` and `ltrsyn` are discussed. The command `lqr` is part of the Control Toolbox while `ltrsyn` is contained in the Robust Toolbox.

Especially in the MIMO case the singular values — presented in section 4.3 — might be useful in the design process.

The command `lqr` can be used to find  $K$  as well as  $H$ :

- $K = \text{lqr}(A, B, \text{rho\_k} * C' * C, \text{eye}(m))$  for a system with  $m$  input signals
- $H = \text{lqr}(A', C', \text{rho\_h} * B * B', \text{eye}(p))'$  for a system with  $p$  output signals

The two steps can be carried out in arbitrary order. For the first step, some different weighting factors can be chosen. For the second step, the special ‘recovery choice’ is recommended, including a large enough value of `rho_h` or `rho_k`, resp.

## 4.2 LTR using MATLAB: Command `ltrsyn`

The steps are as follows:

1. Design of LQ-regulator
2. Design the loop transfer recovery using  $K$  (from LQ-regulator) and the weighting factor  $\rho$

The result of the `ltrsyn` routine is the complete compensator, which contains  $K$  (state controller) as well as  $H$  (observer feedback).

## 4.3 Using Singular Values for LQG/LTR Design

While performing the commands sketched in the two previous sections, it might be advantageous to watch e.g. the singular values of the open loop transfer function. In [2], an appropriate method is described in detail.

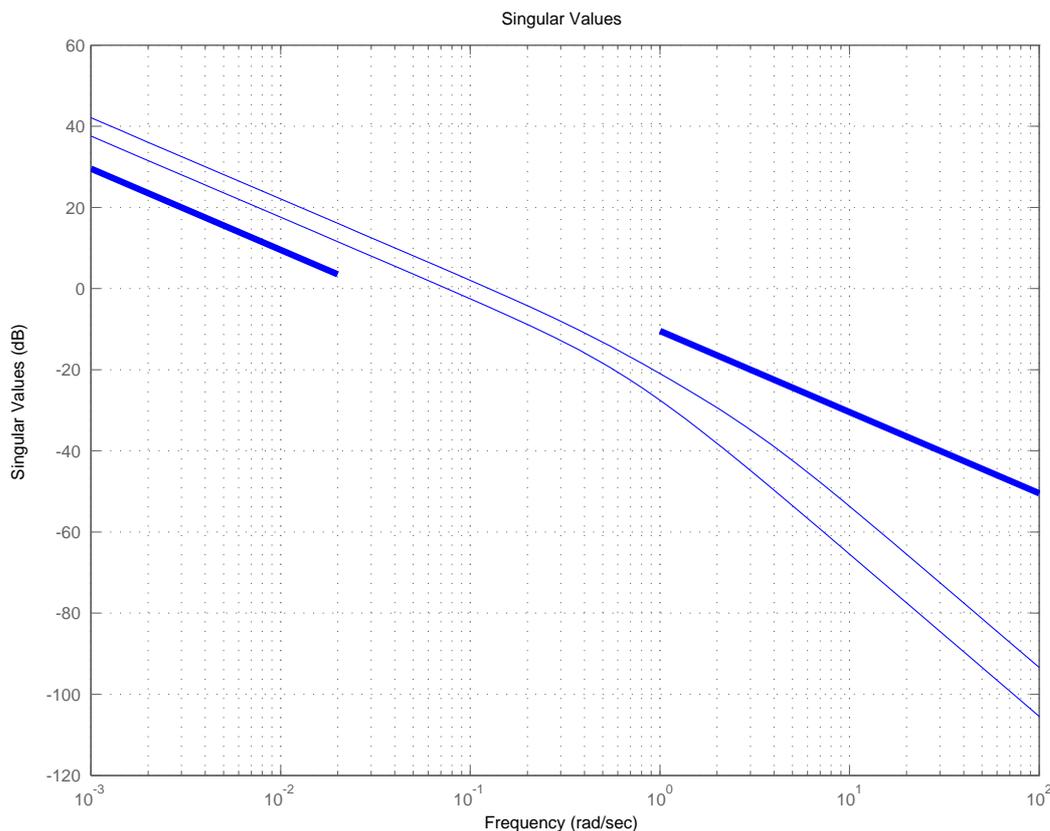


Figure 4: For a MIMO system  $G(s)$ , the command `sigma(G)` produces a plot with largest and smallest singular values,  $\bar{\sigma}$  and  $\underline{\sigma}$ , resp. Using lower and upper bounds in the pass-band and the stop-band is useful to shape the loop transfer.

# Bibliography

- [1] William S. Levine, editor. *The Control Handbook*. The Electrical Engineering Handbook Series. CRC Press/IEEE Press, 1996.
- [2] H. P. Geering. *Regelungstechnik*. Springer, 2004.
- [3] J. C. Doyle. Guaranteed margins for lqr regulators. *IEEE Trans. on Automatic Control*, 1978.



# Appendix A

## Exercises

*Exercise 1.* Design of an observer

Given is a system in state space description

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where only the state  $y = x_1$  can be measured directly.

1. Is the system observable?
2. Design an observer with the characteristic equation  $s^2 + 2\zeta\omega_n s + \omega_n^2$  with  $\zeta = 1.0$  and  $\omega_n = 6$ .
3. Simulate the response of the error  $\tilde{x}$  with the initial condition  $\tilde{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

*Exercise 2.* Similarity transformation

Given is the block diagram in figure A.1 of a feedback system.

1. Find the state-space description for the system. Take  $r$  as input vector,  $x_{tot} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$  as state vector and  $y$  as output vector.
2. Find a transformation matrix  $T$  which transforms the state vector  $x_{tot} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$  into a new state vector  $x_{tot,new} = \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix} = T \cdot \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$ .
3. Using the transformation matrix  $T$ , find the alternative state space description with  $r$  as input vector,  $x_{tot,new}$  as state vector and  $y$  as output vector.  
Hint: in that special case we have  $T^{-1} = T$ .
4. Sketch the block diagram for the description resulting in 3.

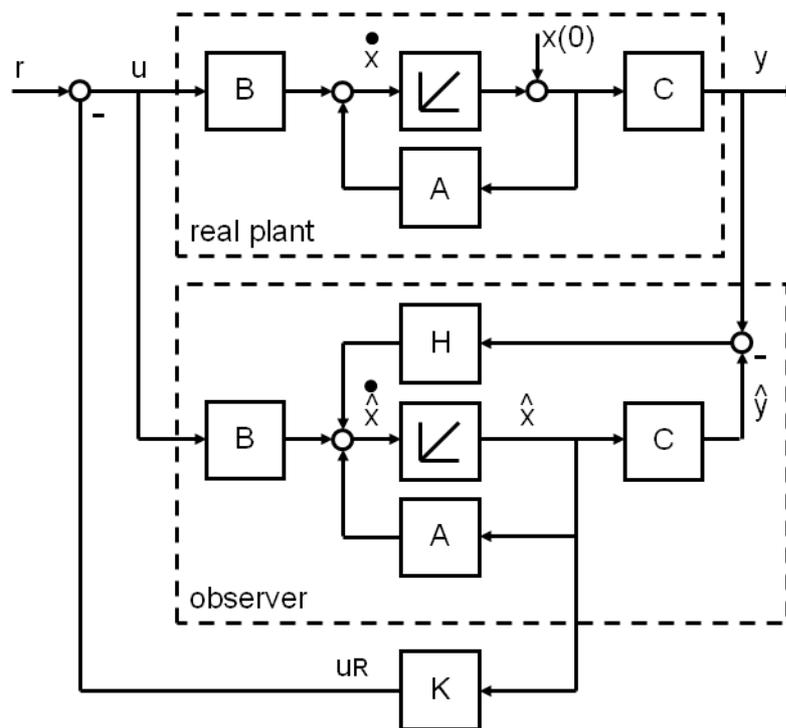


Figure A.1: Plant with observer and state feedback

5. Is the system controllable?

*Exercise 3.* LQG-LTR design with MATLAB

Given is a system

$$A = \begin{bmatrix} -111 & -34.7 & -7.81 \\ 32 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 16 \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \ 0 \ 24.4], \quad D = 0.$$

1. Find an LQR feedback gain  $K$  using  $Q = C^T C$  and  $R = 1$  (with `lqr`).
2. Use the dual LQR design (`lqr`) to recover the loop transfer for some different values of  $\rho$ . Calculate the corresponding observer gains  $H$  using the weights  $Q = \rho B B^T$  and  $R = 1$ . Plot Nyquist diagrams to illustrate the recovery.

# Appendix B

## Solutions of Exercises

*Solution of Exercise 1.* Design of an observer

1.

$$P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$\det(P_o) = 3$$

$\Rightarrow$  observable

2.

$$\begin{aligned} \det(\lambda I - A + HC) &= \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \\ \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} h_1 & 0 \\ h_2 & 0 \end{bmatrix}\right) &= \\ \det\left(\begin{bmatrix} \lambda - 2 + h_1 & -3 \\ 1 + h_2 & \lambda - 4 \end{bmatrix}\right) &= \\ (\lambda - 2 + h_1)(\lambda - 4) - (1 + h_2)(-3) &= \\ \lambda^2 + (h_1 - 6)\lambda + (11 - 4h_1 + 3h_2) &= \lambda^2 + 12\lambda + 36 \\ \Downarrow & \\ h_1 - 6 = 12 \quad \Rightarrow \quad h_1 = 18 & \\ 11 - 4h_1 + 3h_2 = 36 \quad \Rightarrow \quad h_2 = 32\frac{1}{3} & \\ \Rightarrow \quad H = \begin{bmatrix} 18 \\ 32\frac{1}{3} \end{bmatrix} & \end{aligned}$$

3.

Listing B.1: Simulate observer with initial conditions

A = [2 3; -1 4];
B = [0; 1];

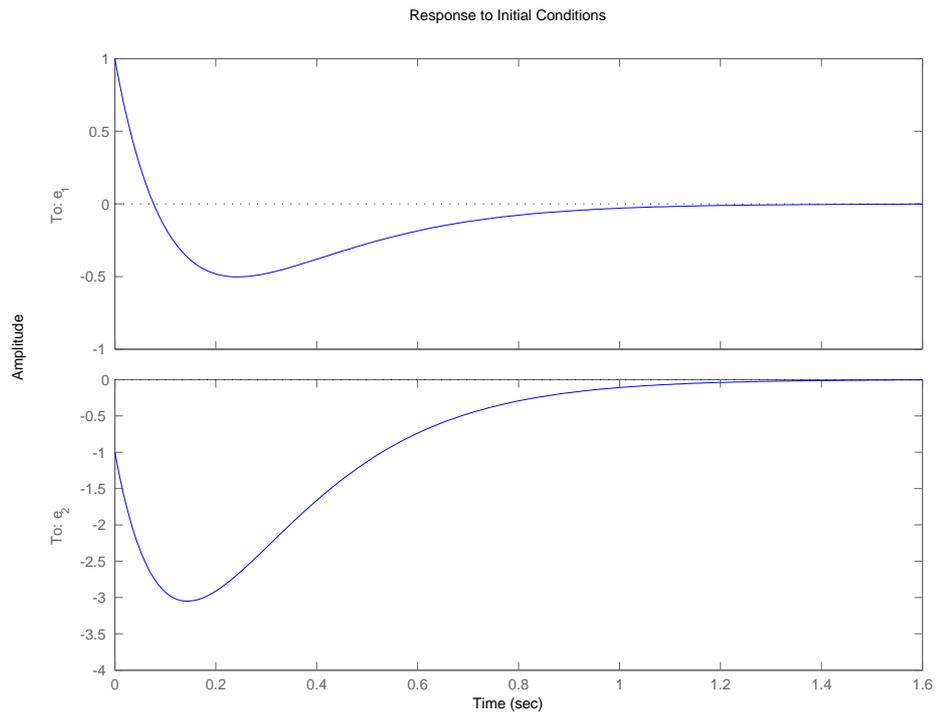


Figure B.1: Response of the observer to an initial condition

```

C = [1 0];
D = 0;
x0 = [1 -1];

H = [18; 32+1/3];
G = ss(A-H*C,B,eye(2),[],'Outputname',{ 'e_1', 'e_2' });

initial(G,x0);

```

Refer to figure B.1 for the plot.

### Solution of Exercise 2. Similarity transformation

1. Since

$$\dot{x} = Ax + Br - BK\hat{x}$$

$$\dot{\hat{x}} = A\hat{x} + Br - BK\hat{x} + HCx - HC\hat{x}$$

$$y = Cx$$

we get

$$\dot{x}_{tot} = \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \overbrace{\begin{bmatrix} A & -BK \\ HC & A - BK - HC \end{bmatrix}}^{A_{tot}} \cdot \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \overbrace{\begin{bmatrix} B \\ B \end{bmatrix}}^{B_{tot}} \cdot r$$

$$y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C_{tot}} \cdot \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \underbrace{0}_{D_{tot}} \cdot r$$

2.

$$T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$$

3.

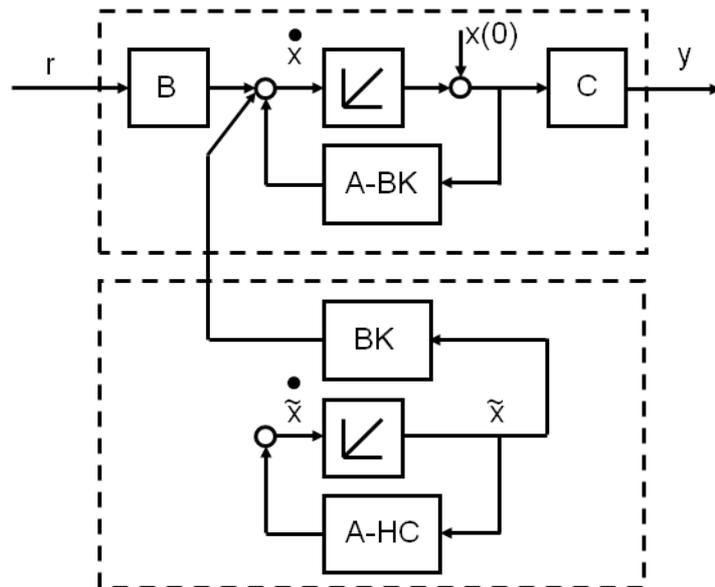
$$A_{tot,new} = T \cdot A_{tot} \cdot T^{-1} = \begin{bmatrix} A - BK & BK \\ 0 & A - HC \end{bmatrix}$$

$$B_{tot,new} = T \cdot B_{tot} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$C_{tot,new} = C_{tot} \cdot T^{-1} = \begin{bmatrix} C & 0 \end{bmatrix}$$

$$D_{tot,new} = D_{tot} = 0$$

4.



5. The system is not controllable — that's obvious by inspection of the block diagram as well as by building the matrix  $P_c = [B_{tot} \ A_{tot}B_{tot} \ \dots \ A_{tot}^{n-1}B_{tot}]$ .

### *Solution of Exercise 3.* LQG-LTR design with MATLAB

1.

Listing B.2: Calculate feedback gain  $K$  using LQR

```
A = [-111 -34.7 -7.81;
      32  0  0;
      0  4  0];
```

```

B = [16; 0; 0];
C = [0 0 24.4];
G = ss(A,B,C,[]);

Q = C'*C;
R = 1;
K = lqr(A,B,Q,R);

```

2.

Listing B.3: Calculate observer gain  $H$  using the LTR technique

```

R = 1;
rho = 10.^(1:2:7);
L = cell(1,length(rho)+1);
L{end} = ss(A,B,K,[]);

for i = 1:length(rho)
    Q = rho(i)*B*B';
    H = lqr(A',C',Q,R)';
    L{i} = ss(A-B*K-H*C,H,K,[])*G;
end
phi = (0:100)*2*pi/100;
nyquist(L{:});
hold on;
plot(cos(phi)-1, sin(phi), 'k');
hold off;
axis([-1 0.5 -1 1]);
axis equal;
legends = cell(1,length(rho)+1);
for i = 1:length(rho)
    legends{i} = ['\rho = ' num2str(rho(i),'%g')];
end
legends{end} = 'LQR';
legend(legends, 'Location', 'East');

```

The Nyquist plots in figure B.2 show that the loop transfer comes closer and closer to the original LQR design the higher  $\rho$  is chosen.

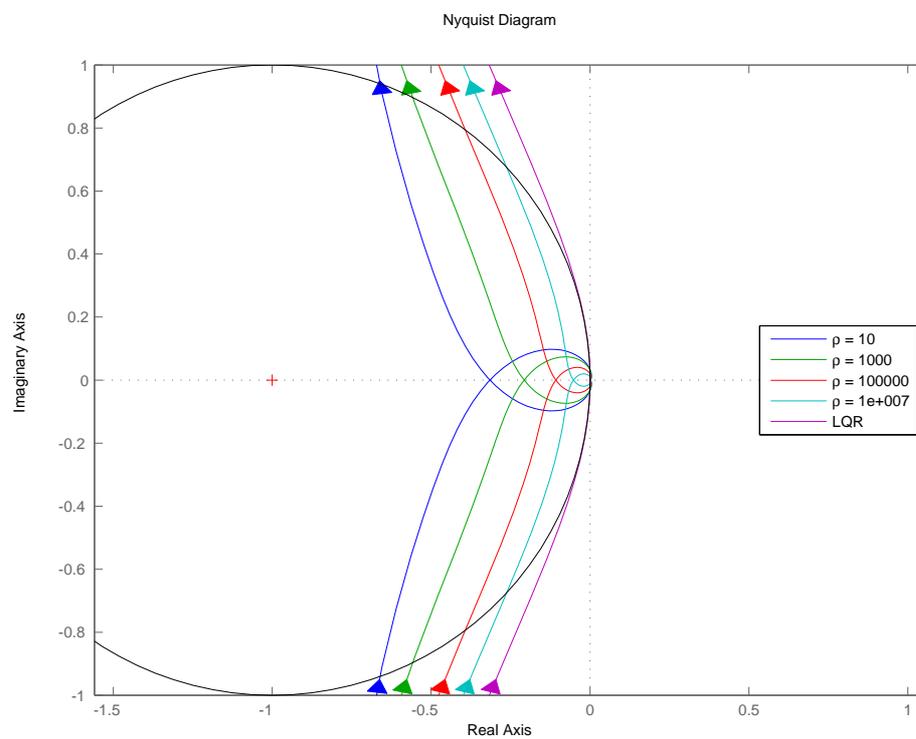


Figure B.2: Nyquist plot of different loop recoveries.