MASTER OF SCIENCE
IN ENGINEERING

Advanced Control

SISO-MIMO Design Examples

Scope	SISO-/ MIMO design examples
Keywords	SISO (Single In - Single Out), MIMO (Multiple In Multiple Out)
Prerequisites	state space description, state regulator (LQR), observer (LQG) and loop transfer recovery (LTR)
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Solved Exercises

1 State Controller including Integral Part

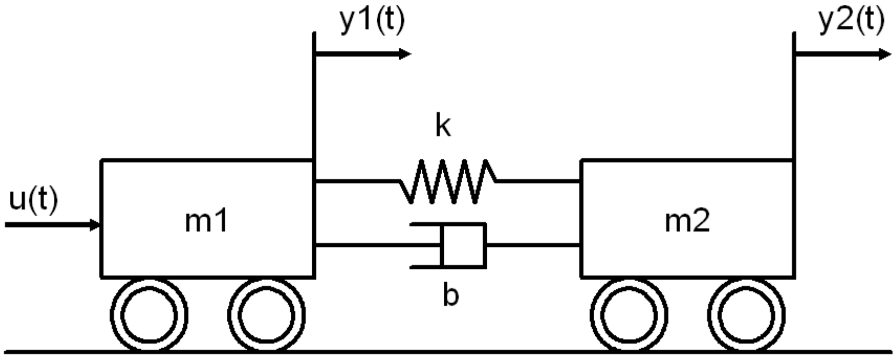
Given is a system in input-output-description

$$Y(s) = \frac{1}{s^2 + 2 \cdot s + 2} \cdot U(s)$$

1. Find the state space description in controllable canonical form.
2. Design a state feedback controller with an integral part and closed loop poles $p_{1,2} = -1 \pm i$, $p_3 = -10$ using MATLAB.
3. Plot the step response of the output y and the error e using MATLAB or Simulink.

2 Two-Mass-Swinger

Given is the system below, with the force u as input signal, and the two positions y_1 and y_2 as output signals.



Assume that $m_1 = 1kg$, $m_2 = 2kg$, $k = 36N/m$, $b = 0.6Ns/m$. The equations for the system are

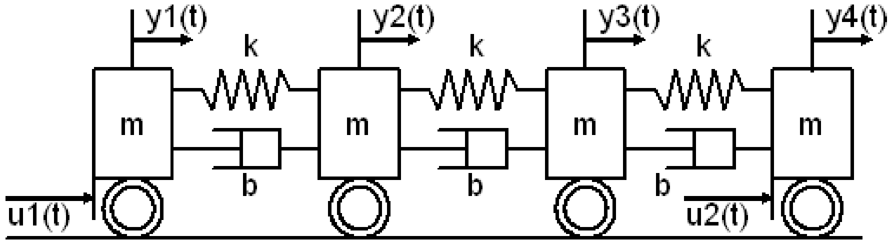
$$m_1\ddot{y}_1 = k(y_2 - y_1) + b(\dot{y}_2 - \dot{y}_1) + u$$
$$m_2\ddot{y}_2 = k(y_1 - y_2) + b(\dot{y}_1 - \dot{y}_2)$$

The state variables are $x_1 = y_1$, $x_2 = y_2$, $x_3 = \dot{y}_1$, $x_4 = \dot{y}_2$.

1. Find a state space description of this system with $y = [y_1 \quad y_2]^T$.
2. Design a state regulator with closed loop poles $p_{1,2} = -2 \pm i \cdot 2\sqrt{3}$, $p_3 = -10$, $p_4 = -10$. Design an observer with poles [-15, -15, -16, -16]. Find a suitable prefilter K_{vf}.
3. Create the closed loop system with the state vector $[x \quad \tilde{x}]^T$ and verify that the eigenvalues of the system correspond to the two pole placement designs.
4. Plot the step response of the outputs y_1 and y_2 and verify the correct selection of K_{vf}.
5. Compare the system responses for the condition when the two mass swinger has a state vector $x = [1\ 1\ 0\ 0]^T$ and the observer $\tilde{x} = \left[0\ 0\ 0\ 0\right]^T$ respective $\tilde{x} = \left[-1\ -1\ 0\ 0\right]^T$. Explain.

3 Four-Mass-Swinger

Given is the system below, with two forces as input, $u = [u_1 \quad u_2]^T$, and four positions as output: $y = [y_1 \quad y_2 \quad y_3 \quad y_4]^T$. The state variables are $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, $x_4 = y_4$, $x_5 = \dot{y}_1$, $x_6 = \dot{y}_2$, $x_7 = \dot{y}_3$, $x_8 = \dot{y}_4$. Assume that $m = 1kg$, $k = 36N/m$, $b = 0.6Ns/m$.

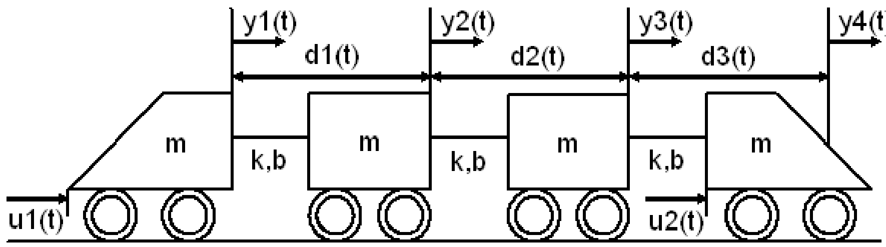


1. Find the state space description of this system.
2. Design a state regulator with closed loop poles $[-2 \pm i \cdot 3, -2 \pm i \cdot 4, -3 \pm i \cdot 3, -3 \pm i \cdot 4]$. Design an observer with poles [-15, -15, -15, -15, -16, -16, -16, -16].
3. Compare the behaviour of the system without and with observer (LQR vs. LQG-LTR). Compare the responses of initial values [-1 -1 -1 -1 0 0 0 0]

   (Initial state of the observer!).
4. Design the state regulator and the observer using optimal control.

4 Train

Given is the system below, where the forces u_1 and u_2 are the inputs and $d_1 = y_2 - y_1$, $d_2 = y_3 - y_2$ and $d_3 = y_4 - y_3$ are the differences between the positions of the waggons. Assume that $m = 1kg$, $k = 36N/m$, $b = 0.6Ns/m$.



The state variables are $x_1 = d_1$, $x_2 = d_2$, $x_3 = d_3$, $x_4 = \dot{d}_1$, $x_5 = \dot{d}_2$, $x_6 = \dot{d}_3$, $x_7 = \dot{y}_1$.

1. Find the state space description for this system with one output $\dot{y}_1$.

2. Design a state regulator with closed loop poles $[-2 \pm i \cdot 3,\ -2 \pm i \cdot 4,\ -3 \pm i \cdot 3,$ -10]. Design an observer with poles [-20, -20, -20, -20, -21, -21, -21].

3. Compare the behaviour of the system without and with observer (LQR vs. LQG-LTR).

4. Plot the step response of the output $\dot{y}_1$. Find a suitable prefilter K_{vf}.

5. Design the state regulator and the observer using optimal control.

Solutions

1 State Controller including Integral Part

1.

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x
\end{aligned}
$$

2.

$$
\begin{bmatrix} \dot{e} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \end{bmatrix}}_{\hat{A}} \begin{bmatrix} e \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}}_{\hat{B}} w
$$

Listing 1: Calculate feedback gain with integral action by pole placing

```
A = [0 1; -2 -2];
B = [0; 1];
C = [1 0];

Ahat = [0 C; zeros(2,1) A];
Bhat = [0; B];

Khat = place(Ahat, Bhat, [-1+1j -1-1j -10]);
K1 = Khat(1);
K2 = Khat(2:end);
```

$$
\begin{aligned}
\hat{K} &= \begin{bmatrix} 20 & 20 & 10 \end{bmatrix} \\
K_1 &= 20 \\
K_2 &= \begin{bmatrix} 20 & 10 \end{bmatrix}
\end{aligned}
$$

3. The system $G_1(s)$ without the integral action is:

$$
\begin{aligned}
\dot{x} &= [A - BK_2]x + Bu \\
y &= Cx
\end{aligned}
$$

The integrator $G_I(s) = K_1 \frac{1}{s}$ is connected in series to $G_1(s)$ to get the loop transfer function $L(s) = G_1(s)G_I(s)$. The transfer function $G(s)$ from r to y is then found by closing the loop. The error is then defined as $e = y - r$ or in laplace $E(s) = Y(s) - R(s) \Leftrightarrow E(s) = G(s)R(s) - R(s) \Leftrightarrow G_e(s) = \frac{E(s)}{R(s)} = G(s) - 1$.

Listing 2: Create closed loop system and plot step response

```
G1 = ss(A-B*K2,B,C,[]);
GI = tf(K1,[1 0]);
L = G1*GI;
G = feedback(L,1);
Ge = G-1;
step(G,Ge);
legend('y','e');
```

Refer to figure 1 for the step response.

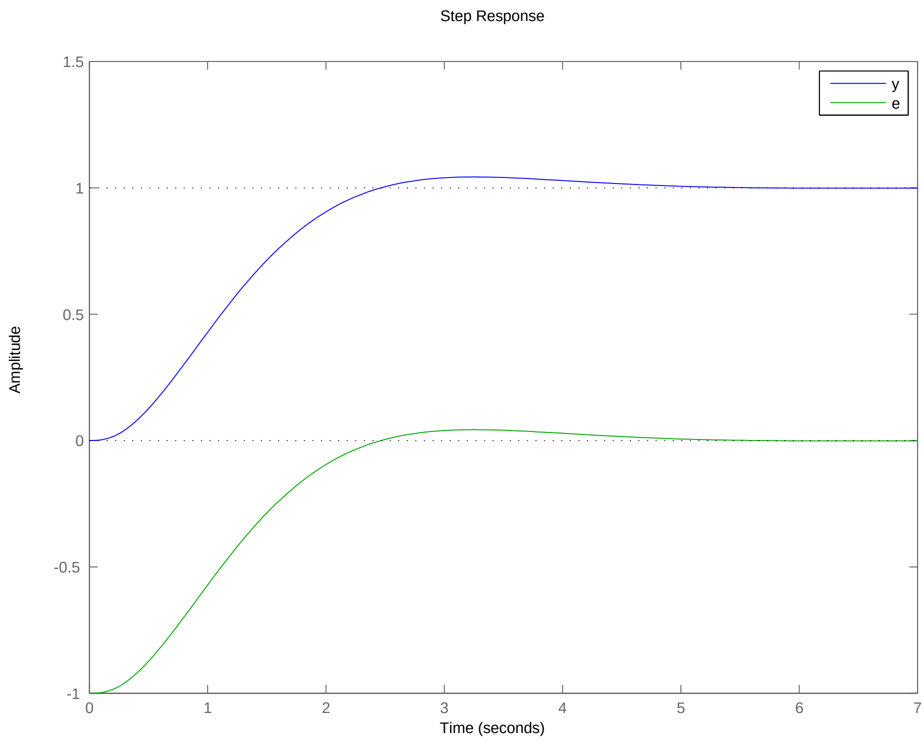


Figure 1: Step response of the state controller with integral action

When you simulate the step response with Simulink you can refer to the block diagram in the script. To verify your implementation use the MATLAB command `linmod('yourSimulinkFile')` to get a system description. Check the eigenvalues of this system to make sure that they correspond to the pole placement design.

2 Two-Mass-Swinger

1.

$$
\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{k}{m_1}x_1 + \frac{k}{m_1}x_2 - \frac{b}{m_1}x_3 + \frac{b}{m_1}x_4 + u \\
\dot{x}_4 &= \frac{k}{m_2}x_1 - \frac{k}{m_2}x_2 + \frac{b}{m_2}x_3 - \frac{b}{m_2}x_4
\end{aligned}
$$

$$
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{k}{m_1} & \frac{k}{m_1} & -\frac{b}{m_1} & \frac{b}{m_1}\\ \frac{k}{m_2} & -\frac{k}{m_2} & \frac{b}{m_2} & -\frac{b}{m_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}u
$$

$$
\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}
$$

2. The design is straight forward. The only thing that needs special attention is the design of the prefilter K_{vf}. Because we have two outputs but only one input, we have to decide for which output we want the static gain to be 1. We select the second output y_2 (in this example the choice does not really matter because both outputs have the same static gain). The static gain is not influenced by the observer because at steady state the observer errors are zero for all suitable H.

Listing 3: Design state regulator and observer by pole placing

```
m1 = 1;
m2 = 2;
k = 36;
b = 0.6;

A = [0 0 1 0; 0 0 0 1; [-k k -b b]/m1; [k -k b -b]/m2];
B = [0; 0; 1; 0];
C = [1 0 0 0; 0 1 0 0];

K = acker(A, B, [-2+2i*sqrt(3) -2-2i*sqrt(3) -10 -10]);
H = place(A', C', [-15 -15 -16 -16])';

Kvf = 1/(C(2,:)/(-A+B*K)*B);
```

$$K = \begin{bmatrix} 130.4444 & -41.5556 & 23.1 & 15.4185 \end{bmatrix}$$
$$H = \begin{bmatrix} 30.4 & 0.6 \\ 0.3 & 30.7 \\ 185.94 & 54.06 \\ 27.03 & 212.97 \end{bmatrix}$$
$$K_{vf} = 88.8889$$

3.

Listing 4: Create the closed loop system and check the eigenvalues

```
Ag = [A-B*K B*K; zeros(size(A)) A-H*C];
Bg = [B; zeros(size(B))];
Cg = [C zeros(size(C))];

G = ss(Ag, Bg, Cg, [])*Kvf;

eig(G)
```

4.

Listing 5: Step response of the closed loop system

```
step(G);
```

Refer to figure 2 for the step response plot.

5.

Listing 6: Initial responses of the closed loop system

```
x0 = [1 1 0 0 0 0 0 0];
initial(G,x0);
hold on
x0 = [1 1 0 0 -1 -1 0 0];
initial(G,x0);
hold off
legend('Observer state [0 0 0 0]', 'Observer state [-1 -1 0 0]'
    )
```

Refer to figure 3 for the initial responses. Because the controller uses inaccurate state estimations during the transients of the observer, the performance of the controller is initially reduced.

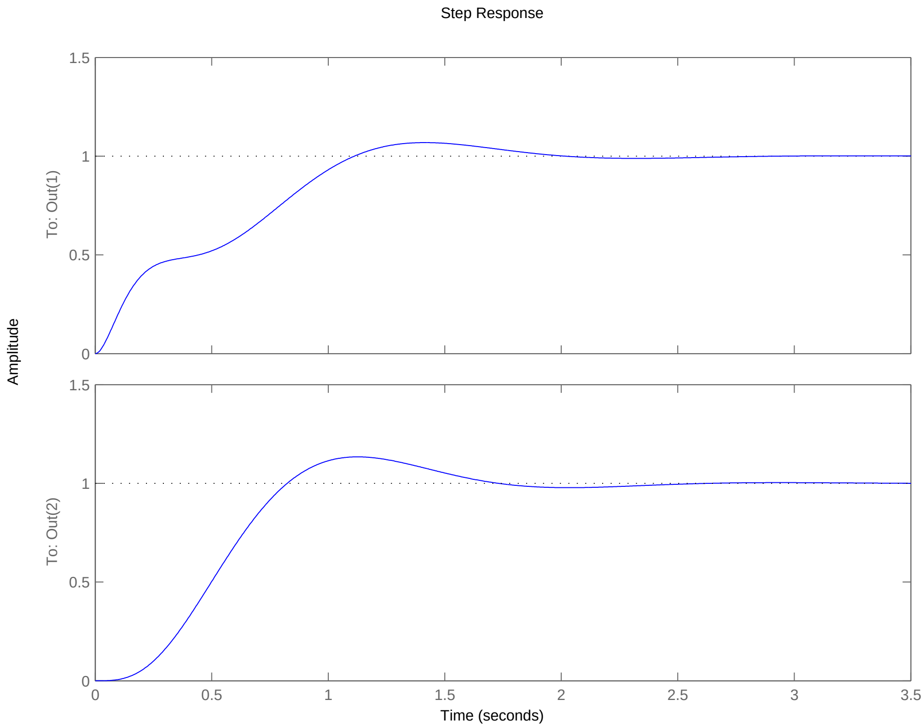


Figure 2: Step response of the closed loop system

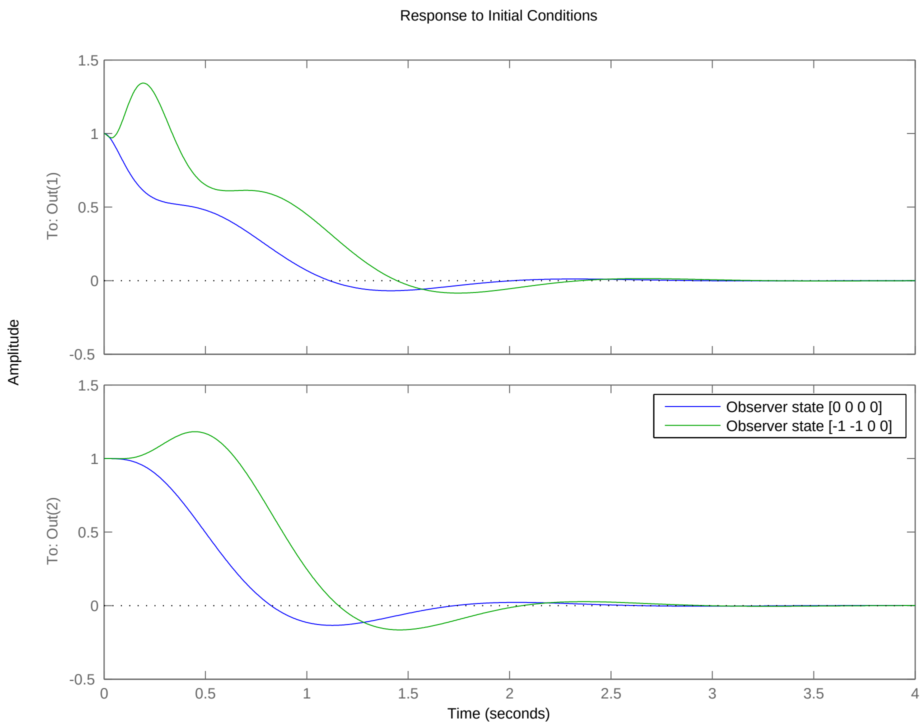
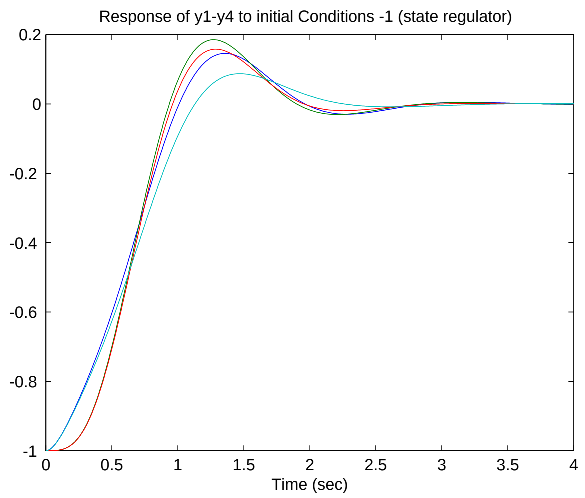


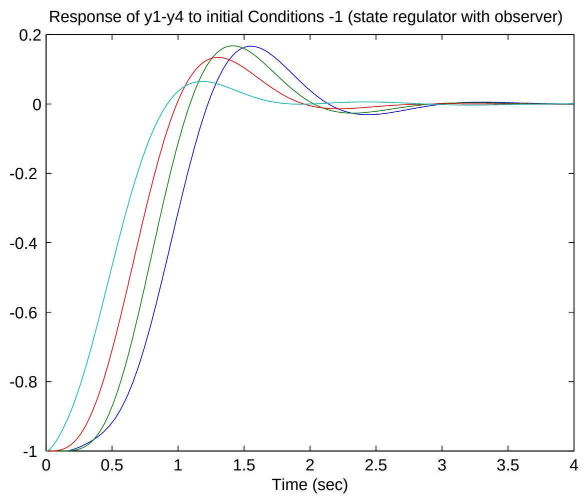
Figure 3: Initial responses of the closed loop system

3 Four-Mass-Swinger

$$m\ddot{y}_1 = k(y_2-y_1) + b(\dot{y}_2-\dot{y}_1) + u_1$$
$$m\ddot{y}_2 = -k(y_2-y_1) - b(\dot{y}_2-\dot{y}_1) + k(y_3-y_2) + b(\dot{y}_3-\dot{y}_2)$$
$$m\ddot{y}_3 = -k(y_3-y_2) - b(\dot{y}_3-\dot{y}_2) + k(y_4-y_3) + b(\dot{y}_4-\dot{y}_3)$$
$$m\ddot{y}_4 = -k(y_4-y_3) - b(\dot{y}_4-\dot{y}_3) + u_2$$

```
A = [0 0 0 0 1 0 0 0;0 0 0 0 0 1 0 0;...
    0 0 0 0 0 0 1 0;0 0 0 0 0 0 0 1;...
    -36 36 0 0 -0.6 0.6 0 0;...
    36 -72 36 0 0.6 -1.2 0.6 0;...
    0 36 -72 36 0 0.6 -1.2 0.6;...
    0 0 36 -36 0 0 0.6 -0.6]
B = [0 0;0 0;0 0;0 0;1 0;0 0;0 0;0 1];

%Position y1-y4
C = [1 0 0 0 0 0 0 0;0 1 0 0 0 0 0 0;0 0 1 0 0 0 0 0;0 0 0 1 0 0
0 0];
D = 0

% State Regulator:
J = [-2+j*3 -2-j*3 -2+j*4 -2-j*4 -3+j*3 -3-j*3 -3+j*4 -3-j*4];
K = place (A,B,J);
%K = lqr(A,B,1*C'*C,[0.0001 0;0 0.0001]);

t = 0:0.01:4;
SYSP = ss(A-B*K,B,C,D);
P = initial (SYSP,[-1;-1;-1;-1;0;0;0;0],t);
figure(1); plot(t,P); axis ([0 4 -1 0.2]);

% Observer:
Ph = [-15 -16 -15 -16 -15 -16 -15 -16];
H = place (A',C',Ph');
% H = lqr(A',C',B*B',0.01* eye(4));
H = H';

AA = [A-B*K B*K; zeros(8,8) A-H*C];
BB = [B;zeros(8,2)];
CC = [C zeros(4,8)];
DD = D;

SYSO = ss(AA,BB,CC,DD)
O = initial (SYSO,[-1;-1;-1;-1;0;0;0;0;0.2;0;0;0;0;0;0;0],t);
figure(2); plot(t,O); axis ([0 4 -1 0.2]);
```


Response of y1-y4 to initial Conditions -1 (state regulator)
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.5
1
1.5
2
2.5
3
3.5
4
Time (sec)


Response of y1-y4 to initial Conditions -1 (state regulator with observer)
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.5
1
1.5
2
2.5
3
3.5
4
Time (sec)

4 Train

$$m(\overbrace{\ddot{y}_2-\ddot{y}_1}^{\ddot{d}_1}) = -2kd_1 - 2b\dot{d}_1 + kd_2 + b\dot{d}_2 - u_1$$

$$m(\overbrace{\ddot{y}_3-\ddot{y}_2}^{\ddot{d}_2}) = k\,d_1 + b\dot{d}_1 - 2kd_2 - 2b\dot{d}_2 + kd_3 + b\dot{d}_3$$

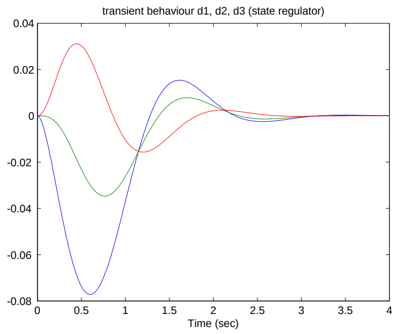
$$m(\overbrace{\ddot{y}_4-\ddot{y}_3}^{\ddot{d}_3}) = kd_2 + b\dot{d}_2 - 2kd_3 - 2b\dot{d}_3 + u_2$$

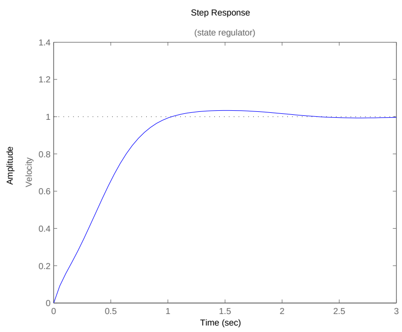
$$m\ddot{y}_1 = k\cdot d_1 + b\dot{d}_1 + u_1$$

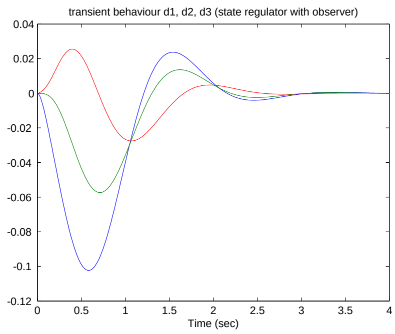
```
A = [0 0 0 1 0 0 0;0 0 0 0 1 0 0;0 0 0 0 0 1 0;...
    -72 36 0 -1.2 0.6 0 0;...
    36 -72 36 0.6 -1.2 0.6 0;...
    0 36 -72 0 0.6 -1.2 0;...
    36 0 0 0.6 0 0 0];
B = [0 0;0 0;0 0;-1 0;0 0;0 1;1 0];
C = [0 0 0 0 0 0 1; 1 0 0 0 0 0 0;0 1 0 0 0 0 0;0 0 1 0 0 0 0];
D = zeros (4,2);
```

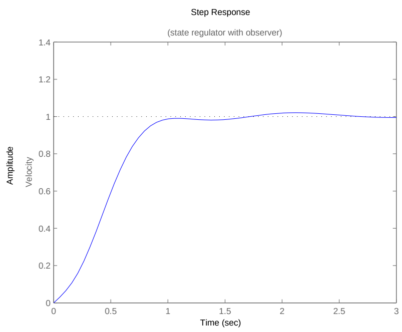
Output → ↖ Inputs

```
J = [-2+j*3 -2-j*3 -2+j*4 -2-j*4 -3+j*3 -3-j*3 -10];
K = place (A,B,J);
%Q = eye(7);
%K = lqr(A,B,Q,[0.01 0;0 0.01]);

t = 0:0.01:4;
SYSP = ss(A-B*K,B,C,D);
P = initial (SYSP,[0;0;0;0;0;0;-1],t);
[numP,denP] = ss2tf (A-B*K,B,C,D,1);
Kvf = denP (8)/numP(1,8);

figure(1);
plot(t,P(:,2:4))
figure(2);
step(Kvf*numP(1,:),denP);

% with observer:
%H = lqr(A',C',B*B',0.1*eye(4));
%H = H';
I = [-20 -21 -20 -21 -20 -21 -20];
H = place(A',C',I)';

AA = [A-B*K B*K; zeros(7,7) A-H*C];
BB = [B;zeros(7,2)];
CC = [C zeros(4,7)];
DD = zeros (4,2);

SYSP = ss(AA,BB,CC,DD)
O = initial (SYSP,[0;0;0;0;0;0;-1;0.03;0;0;0;0;0;0],t);
[numO,denO] = ss2tf (AA,BB,CC,DD,1);
Kvf = denO (15)/numO(1,15);

figure(3);
plot(t,O(:,2:4));
figure(4);
step(Kvf*numO(1,:),denO);
```


transient behaviour d1, d2, d3 (state regulator)
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
0
0.5
1
1.5
2
2.5
3
3.5
4
Time (sec)


Step Response
(state regulator)
Amplitude
Velocity
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
Time (sec)


transient behaviour d1, d2, d3 (state regulator with observer)
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
0
0.5
1
1.5
2
2.5
3
3.5
4
Time (sec)


Step Response
(state regulator with observer)
Amplitude
Velocity
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
Time (sec)